# DREXEL INSTITUTE OF TECHNOLOGY

# METHODS FOR SYSTEMATIC

## GENERATION OF

## LIAPUNOV FUNCTIONS (PART TWO)

Submitted by

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CONTRACT NO. NAS 8-11196

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## ACKNOWLEDGMENTS

ABSTRACT

This report summarizes much of the work that has done in the field of stability theory with regards to the generation of Liapunov functions.  The emphasis of the report has been to survey and discuss the work of American engineers and mathematicians in this area.  But since most of the work was motivated by Russian mathematicians and engineers, this report also includes a sizable discussion of the Russian contributions.  Reference is also made to the contributions due to mathematicians in England, Japan and Italy.  Under separate cover, the writers of this report submit a sizable list of references in the stability field and a summary of the theorems and definitions which are important in the analysis of stability problems.

## LIST OF SYMBOLS

(most symbols are defined where they are used in the report and will not be repeated here)

$V$ = usually denotes a scalar function, or functional which is a Liapunov function or a candidate for a Liapunov function.

$\| \underline{x} \|$ = usually denotes the Euclidean norm of an n-dimensional vector, defined as:

$$\left[ x_1^2 + \ldots + x_n^2 \right]^{1/2}.$$

$t \in [a, b]$      means $a \leq t \leq b$.

$t \in (a, b]$      means $a < t \leq b$.

$t \in (a, b)$      means $a < t < b$.

$a \in A$      means that element $a$ is a member of set $A$.

$\underline{A}_T$ = transpose of matrix $\underline{A}$

$\underline{A}^*$ = conjugate transpose of matrix $\underline{A}$.

$\underline{\dot{x}}$ = time derivative of the vector function, $\underline{x} \equiv \underline{x}(t)$.

$E_n$ = Euclidean n-space.

$C^n$ = The class of functions having continuous n-th order partial derivatives.

$\underline{\nabla V}$ = gradient of the scalar function $V$.

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## WORK OF KRASOVSKII, MANGASARIAN, CHANG, INGWERSON AND SZEGO

### SUMMARY

This section considers two important theorems of Krasovskii and the generalizations which were considered by others, based on Krasovskii's work. We present the work of Mangasarian, the kinetic Liapunov function of Chang, the modified Liapunov theory of Ingwerson and the "generating V - function" of Szego. Also, we include a compendium of examples at the end of the section.

### INTRODUCTION

In 1954, Krasovskii published an important result concerning the global stability of a system of differential equations. This result in itself was not as significant as the work which Krasovskii motivated throughout the "differential equation community". Krasovskii's theorem dealt with autonomous differential equations whose right sides were continuously differentiable.

In 1957, Krasovskii considered nonautonomous systems with right sides which were continuously differentiable and with large initial disturbances.

Chang, in 1961, introduced his kinetic Liapunov function which is a Liapunov function of the first derivatives of the state variable. This formulation had been anticipated by Krasovskii's work. The use of this function leads to sufficient conditions for asymptotic stability in the large for nonlinear, nonautonomous systems.

In 1963, Mangasarian extended Krasovskii's work to cover nonautonomous, nondifferentiable cases. In this work, the properties of convex functions were used in determining stability requirements.

Ingwerson's work, 1961, modified the original Liapunov stability criterion. His method of generating Liapunov functions consisted of solving a matrix equation in closed form, modifying the result, and then performing a type of double integration. His method is well motivated, can often be carried out analytically and often gives excellent results.

In 1962, Szegö considered the stability of nonlinear autonomous systems with the nonlinearity representable in polynomial form. Szegö considers "generating V - functions" of the form $V(\underline{x}) = \underline{x}_T \underline{A}(\underline{x})\underline{x}$, where $\underline{A}$ is a symmetric matrix function of the state variable. The elements of $\underline{A}$, $a_{ij}(x_i, x_j)$, can be computed in such a way as to obtain $\dot{V}$ of the wanted form. Particular attention is given to studying limit cycles of systems.

## KRASOVSKII'S WORK

From references $[1,2]$ we have the following statement of Krasovskii's Theorem. This theorem considers a free, stationary, dynamic system described by

$$\dot{\underline{x}} = \underline{f}(x),$$

$$f(\underline{0}) = \underline{0}, \tag{1}$$

where $\underline{f}$ has continuous first partial derivatives. Also, we define $\underline{F} = \underline{F}(\underline{x})$ to be the Jacobian matrix of $\underline{f}(\underline{x})$; thus, $\underline{F} = \left[ \dfrac{\partial f_i(\underline{x})}{\partial x_j} \right]$.

## Theorem

H) (i) Equations (1) describe a free, stationary, dynamic system,

(ii) $\underline{f}$ has continuous first partial derivations,

(iii) $\hat{\underline{F}}(\underline{x}) = \underline{F} + \underline{F}_T \leq - \in \underline{I}$ for any $\in > 0$ in a neighborhood of $\underline{x} = \underline{0}$,

C) then $\underline{x} = \underline{0}$ is asymptotically stable in the large.

## Proof

The candidate which is chosen for a Liapunov function is $V = \underline{f}_T\,\underline{f}$. The corresponding time derivative of $V$ is

$$\dot{V} = \dot{\underline{f}}_T\underline{f} + \underline{f}_T\dot{\underline{f}} = (\underline{F}\,\dot{\underline{x}})_T\,\underline{f} + \underline{f}_T\,(\underline{F}\,\dot{\underline{x}})$$

$$= \underline{f}_T\,(\underline{F}_T + \underline{F})\underline{f} = \underline{f}_T\,\hat{\underline{F}}\,\underline{f} \leq - \in \underline{f}_T\underline{f}\ .$$

By hypothesis (ii), we know that a Taylor's expansion of $\underline{f}_T\underline{f}$ in some neighborhood of $\underline{x} = \underline{0}$ gives

$$\dot{V} \leq - \in \underline{f}_T\underline{f} \leq - \in \|\underline{x}\|^2 \leq 0\ .$$

It remains to be shown that $V$ is positive definite and tends to infinity with

$$\|\underline{x}\| \longrightarrow \infty\ .$$

Let $\underline{c}$ be a constant vector, $\underline{c} \neq \underline{0}$. The set of vectors $\left\{\alpha\underline{c};\ 0 \leq \alpha \leq 1\right\}$ is a ray connecting the origin with $\underline{c}$. Integrating along the ray, we have

$$f_i(\underline{c}) = \sum_{j=1}^{n}\left[\int_0^1 c_j\ \frac{\partial f_i(\alpha\underline{c})}{\partial x_j}\,d\alpha\right]\ .$$

Suppose that $\underline{f}(\underline{c})$ is $\underline{0}$ for some $\underline{c} \neq \underline{0}$. Then

$$0 = \underline{c}_T\,\underline{f}(\underline{c}) = \sum_{i=1}^{n} c_i f_i(\underline{c}) = \int_0^1 \sum_{i,j=1}^{n} c_i c_j\ \frac{\partial f_i(\alpha\underline{c})}{\partial x_j}\,d\alpha$$

$$\leq - \frac{\in \|\underline{c}\|^2}{2} < 0\ ,$$

which is a contradiction. Hence $V = \underline{f}_T\underline{f}$ is positive definite since $\underline{f} = \underline{0}$ if and only if $\underline{x} = \underline{0}$. Also, the above argument shows that

$$\underline{x}_T\,\underline{f}(\alpha\underline{x}) \longrightarrow - \infty \quad \text{with} \quad \alpha \longrightarrow \infty\ , \quad \text{for any fixed}$$

vector $\underline{x} \neq 0$. This means that at least one component of $\underline{f}$ tends to $\infty$ in absolute value as $\|\underline{x}\|$ tends to $\infty$ ; this <u>completes the proof</u> of Krasovskii's theorem.

In the above proof, a more general Liapunov function could be used; such as,

$$V = \underline{f}_T \underline{A} \, \underline{f} \, ,$$

where $\underline{A}$ is a symmetric matrix. The time derivative of this V is

$$\dot{V} = \underline{f}_T \, (\underline{A} \, \underline{F} + \underline{F}_T \, \underline{A}) \, \underline{f} = - \underline{f}_T \, \underline{B} \, \underline{f} \, ,$$

where $\underline{B} = - (\underline{A} \, \underline{F} + \underline{F}_T \, \underline{A})$ is positive definite.

Also, in reference [3], the statement and proof of Krasovskii's theorem is given in slightly different form. We will repeat this in the following paragraph.

<u>Theorem</u>

    H) (i) $\underline{f}$ in equation (1) has continuous first partials,

       (ii) $\hat{F}$ has negative eigenvalues for all $\underline{x}$,

    C) $\underline{x} = \underline{0}$ is completely stable.

<u>Proof</u>

We take as a candidate for a Liapunov function, $V = \underline{x}_T \underline{x}$; thus, as $\|\underline{x}\|^2 \longrightarrow \infty$ , $V \longrightarrow \infty$ . The time derivative of V is

$$\dot{V} = \dot{\underline{x}}_T \underline{x} + \underline{x}_T \dot{\underline{x}} = \underline{f}_T(\underline{x})\underline{x} + \underline{x}_T f(\underline{x}).$$

The scalar product of $\underline{x}$ with $\nabla \dot{V}$ is

$$(\nabla \dot{V} \cdot \underline{x}) = \underline{x}_T \underline{f}(\underline{x}) + \underline{f}_T(\underline{x}) \underline{x} + \underline{x}_T (\underline{F}_T + \underline{F}) \underline{x}$$

$$= \dot{V} + \underline{x}_T (\underline{F}_T + \underline{F}) \underline{x}.$$

Thus,

$$(\nabla \dot{V} \cdot \underline{x}) - \dot{V} = \underline{x}_T (\underline{F}_T + \underline{F}) \underline{x}$$

$$= \underline{x}_T \hat{F} \, \underline{x} < 0,$$

since $\hat{\underline{F}}$ has negative eigenvalues for all $\underline{x}$. Because it can be shown that

$$\nabla \left\{ \frac{\dot{V}}{\sqrt{\|\underline{x}\|^2}} \right\} \cdot \dot{\underline{x}} = \frac{(\nabla \dot{V}) \cdot \dot{\underline{x}} - \dot{V}}{\|\underline{x}\|}$$

we then conclude that the gradient of $\left\{ \dot{V} \Big/ \|\underline{x}\| \right\}$ is always directed toward the origin. Therefore, $\dot{V}$ is negative definite since $\dot{V}$ is of higher order than $\|\underline{x}\|$ and since $\dot{V}(\underline{0}) = 0$. This proves that $\underline{x} = \underline{0}$ is a completely stable equilibrium point.

In summary, we see that Krasovskii's Theorem is limited in application because of the demand that $\underline{f}$ have continuous first partials and that equation (1) describes an autonomous system. But this theorem has influenced the work of others; such as, Ingwerson's work which appears later in this section, Schultz's and Gibson's work which appears in the variable gradient section of this report and Mangasarian's work which will be the next topic of discussion in this section. In the compendium of examples at the end of this section we give some applications of the above theorems.

## MANGASARIAN'S WORK

The main results of Mangasarian's work [6] are the sufficient conditions for the (1) stability, (2) uniform asymptotic stability in the large and (3) instability of the equilibrium point $\underline{x} = \underline{0}$ of the system of differential equations:

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}) \tag{1}$$

$$\underline{f}(t, \underline{0}) = \underline{0}.$$

The sufficient conditions are obtained by using the stability and instability criteria of Liapunov and properties of concave and convex functions. The system given by (1) is an $n^{th}$ order nonlinear system where $0 \leq t < \infty$. According to Massera [7], the stability

criteria and the various modifications of Liapunov theory hold if $\underline{f}$ is

piecewise continuous in $(\underline{x}, t)$ space. The discontinuities of $\underline{f}$ must lie on

sufficiently smooth manifolds; and $\underline{f}$ must be such that for any given $(\underline{x}_o, t)$

there corresponds at least one function of t, $\underline{x} = \underline{y}$ (t, $\underline{x}_o$ , $t_o$), defined,

continuous, and with piecewise continuous derivatives with respect to t

for all t $\geqslant$ $t_o$, which satisfies the equations in (1), except at the

points of discontinuities.

The results are presented in the form of concave and convex scalar

functions, such as,

$$\phi(\underline{x}) = \underline{x}_T \underline{f} \ (t, \underline{x}) \ , \tag{2}$$

where the function $\phi(\underline{x})$ in (2) depends on the n-dimensional vector $\underline{x}$. Before

"going on" with the stability problem, we will consider some of the pro-

perties of concave and convex functions.  For simplicity, we will discuss

some of the results and properties of convex functions of scalar variables

as presented by Beckenbach in $\begin{bmatrix}8\end{bmatrix}$.

A real function f(x), defined in the interval a $<$ x $<$ b, is said to be

convex provided that for all $x_1$ and $x_2$, with a $<$ $x_1$ $<$ $x_2$ $<$ b, and for all

$\lambda$ satisfying $0 \leq \lambda \leq 1$, we have

$$(1-\lambda) \ f \ (x_1) + \lambda f(x_2) \geqslant f \left[(1 -\lambda)x_1 + \lambda x_2\right]. \tag{3}$$

A convex function is necessarily continuous for a $<$ x $<$ b. Geometrically,

the condition of convexity is that each arc of the curve y = f(x) lies

nowhere above the chord joining the end points of the arc.  If $f''(x)$ exists

at each point of the interval, then a necessary and sufficient condition

that f(x) be convex is that $f''(x)$ $>$ 0 for a $<$ x $<$ b. If the strict

inequality in (3) holds throughout, we say that f(x) is strictly convex.

Similar definitions hold for concave functions.  That is, f(x) is concave

if and only if $\left\{ -f(x) \right\}$ is convex.

We will now state some other important properties of convex functions which will be useful in the interpretation of Mangasarian's theorems.

(i)     If $f(x)$ and $g(x)$ are convex functions in the interval $a < x < b$, then $f(x) + g(x)$ and $\max \left\{ f(x), g(x) \right\}$ are convex over $a < x < b$, as is $cf(x)$ if $c \geq 0$.

(ii)    The limit of a convergent sequence of convex functions is convex; also, if it is finite, so is the upper envelope of a family of convex functions.

(iii)   A convex function has a right-hand and left-hand derivative at each point of $(a, b)$. The right-hand is greater than or equal to the left-hand derivative and both are _not_ decreasing functions. These two derivatives at a point are equal except for a denumerable set of points.

(iv)    In the segment of $(a, b)$ outside the subinterval $(x_1, x_2)$, the graph lies nowhere below the line through $\left\{ x_1, f(x_1) \right\}$ and $\left\{ x_2, f(x_2) \right\}$.

(v)     If for fixed $x_1$ and $x_2$ in $(a, b)$ the sign of equality in (3) holds at a single interior point of the subinterval $(x_1, x_2)$, then the sign of equality holds throughout $(x_1, x_2)$.

Some simple _examples_ of convex functions are given below:

(i)     $|x - a|$ is convex; its graph is V-shaped.

(ii)  $g(x) = 2 |x| + |x - 1| + |x - 2|$ is a continuous graph which consists of a succession of line-segments.

(iii)For the $g(x)$ in (ii), the function $\max \left\{ x^2, g(x) \right\}$ is convex.

(iv) By using the derivative test, we can show that for $x > 0$, the functions $\left\{ x \log x \right\}$ and $\left\{ \log \frac{1}{x} \right\}$ are convex.

We now return to the scalar function of vector $\underline{x}$ in (2). It is assumed that $\phi(\underline{x})$ is defined over a convex region. A _convex region_ is a set of points, n-vectors, such that for every pair of points $\underline{x}_1$ and $\underline{x}_2$ in the set, the "line segment" $\underline{x}_1 \underline{x}_2$ is contained in the set. If for all vectors $\underline{x}_1$ and $\underline{x}_2$ in the convex region of definition of $\phi(\underline{x})$, the inequality

$$(1 - \lambda) \phi(\underline{x}_1) + \lambda \phi(\underline{x}_2) \geq \phi\left[(1 - \lambda) \underline{x}_1 + \lambda \underline{x}_2\right] \tag{4}$$

holds for $0 \leq \lambda \leq 1$, then $\phi(\underline{x})$ is called a _convex_ function. The function $\phi(\underline{x})$ is _concave_ if the inequality sign in (4) is reversed. For _strictly convex_ (concave) functions the equality sign in (4) holds only for $\lambda = 0$, $\lambda = 1$, or $x_1 = x_2$. Convexity and concavity imply continuity in the interior of the convex region of definite but not necessarily differentiability. But if $\phi(\underline{x})$ is twice continuously differentiable, then sufficient conditions for convexity, concavity, strict convexity and strict concavity of $\phi(\underline{x})$ are that the symmetric Jacobian matrices,

$$\left[\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right]$$

, be positive semidefinite, negative semidefinite, positive definite and negative definite, respectively for all values of $\underline{x}$ in region of definition of $\phi$.

Since the system in (1) is in general a nonautonomous system, a potential Liapunov function will also be a function of both t and $\underline{x}$. The following definitions, [6], concerning definiteness of $V(t, \underline{x})$ do not assume that V is continuous and has continuous first partial derivatives. The scalar function $V(t, \underline{x})$ is _positive definite_ if for $0 \leq t < \infty$, (i) $V(t, \underline{x}) > 0$ for $\underline{x} \neq \underline{0}$, (ii) $V(t, \underline{0}) = 0$, and (iii) $\lim_{t \to \infty} \inf \left\{V(t, \underline{x})\right\} > 0$ for $\underline{x} \neq \underline{0}$. The function $V(t, \underline{x})$ is _negative definite_ if for $0 \leq t < \infty$, (i) $V(t, \underline{x}) < 0$ for $\underline{x} \neq \underline{0}$, (ii) $V(t, \underline{0}) = 0$, and (iii) $\lim_{t \to \infty} \sup V(t, \underline{x}) < 0$

for $\underline{x} \neq \underline{0}$. The function is said to have an <u>infinitely small upper bound</u> if, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|V| < \epsilon$ for all $t \geq 0$ and $\|\underline{x}\| < \delta$. When V has continuous first partials in t and $\underline{x}$, we have along the solutions of (1)

$$\dot{V}(t,\underline{x}) = \dot{\underline{x}}_T \nabla V + \frac{\partial V}{\partial t} = \underline{f}_T \nabla V + \frac{\partial V}{\partial t}$$

where $\left\{ \nabla V_T = \left( \frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n} \right) \right\}$.

Mangasarian considers three lemmas, 1, 2 and 3, which are used in proving his main theorems 1, 2 and 3, respectively. These lemmas are given below:

<u>Lemma 1.</u> Let $\underline{f}(t, \underline{x})$ be continuous in $\underline{x}$ at $\underline{x} = \underline{0}$ for $0 \leq t < \infty$ and let $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$. If $\underline{x}_T \underline{f}(t, \underline{x})$ is a concave function of $\underline{x}$ for $0 \leq t < \infty$, then $\underline{x}_T \underline{f}(t, \underline{x}) \leq 0$ for $0 \leq t < \infty$.

<u>Lemma 2.</u> Let $\underline{f}(t, \underline{x})$ be continuous in $\underline{x}$ at $\underline{x} = \underline{0}$ for $0 \leq t < \infty$, let $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$ and let $\gamma(\underline{x}) \equiv \lim_{t \to \infty} \sup \left\{ \underline{x}_T \underline{f}(t, \underline{x}) \right\}$. If $\underline{x}_T f(t, \underline{x})$ is a strictly concave function of $\underline{x}$ for $0 \leq t < \infty$ and if either (I) $\gamma(\underline{x}) < 0$ for $\underline{x} \neq \underline{0}$, or (II) $\gamma(\underline{x})$ is strictly concave in $\underline{x}$, then $\underline{x}_T \underline{f}(t, \underline{x})$ is negative definite.

<u>Lemma 3.</u> Let $\underline{f}(t, \underline{x})$ be continuous in $\underline{x}$ at $\underline{x} = \underline{0}$ for $0 \leq t < \infty$, let $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$, and let $\gamma(\underline{x}) \equiv \lim_{t \to \infty} \inf \left\{ \underline{x}_T \underline{f}(t, \underline{x}) \right\}$. If $\underline{x}_T f(t, \underline{x})$ is a strictly convex function of $\underline{x}$ for $0 \leq t < \infty$ and if either (I) $\gamma(\underline{x}) > 0$ for $\underline{x} \neq \underline{0}$, or (II) $\gamma(\underline{x})$ is strictly convex in $\underline{x}$, then $\underline{x}_T \underline{f}(t, \underline{x})$ is positive definite.

Also, Mangasarian makes use of the theorems of Liapunov and Massera. They are:

## Liapunov's Stability Theorem

If a positive definite scalar function $V(t, \underline{x})$ with continuous first partials in t and $\underline{x}$ exists for which $\dot{V} \leq 0$, then the point $\underline{x} = \underline{0}$ is a stable equilibrium point of system (1).

## Massera's Theorem

If a scalar function $V(t, \underline{x})$ with continuous first partials in t and $\underline{x}$ exists which is positive definite, tends to infinity with $\|\underline{x}\|$, has an infinitely small upper bound, and is such that $\dot{V}(t, \underline{x})$ is negative definite, then $\underline{x} = \underline{0}$ is a uniformly, asymptotically stable point in the large of system (1).

## Liapunov's Instability Theorem

If a function $V(t, \underline{x})$ with continuous first partials in t and $\underline{x}$ exists which is positive definite, has an infinitely small upper bound and is such that $\dot{V}$ is positive definite, then $\underline{x} = 0$ is unstable.

We will now state the main theorems of Mangasarian, along with appropriate notes about the proofs.

## Theorem 1 (Stability)

Let $\underline{f}(t, \underline{x})$ be continuous in $\underline{x}$ at $\underline{x} = \underline{0}$ for $0 \leq t < \infty$ and let $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$. If $\underline{x}_T \underline{f}(t, \underline{x})$ is a concave function of $\underline{x}$ for $0 \leq t < \infty$, then the point $\underline{x} = \underline{0}$ is a stable equilibrium point of the system (1).

## Notes About the Proof

A Liapunov function, $V(\underline{x}, t) = \underline{x}_T \underline{x}$, is considered. Thus, $\dot{V} = 2\underline{x}_T \dot{\underline{x}} = 2\underline{x}_T \underline{f}(t, \underline{x})$. From Liapunov's Stability Theorem and the hypotheses in this theorem, the conclusion follows.

## Corollary 1

If $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$ and if the function $\underline{f}(t, \underline{x})$ is a twice continuously differential function of $\underline{x}$ for $0 \leq t < \infty$ and $\| \underline{x} \| < \infty$ , then the point $\underline{x} = \underline{0}$ is a __stable__ equilibrium point of (1) provided that the matrix $\left[ \dfrac{\partial^2 \underline{x}_T \; \underline{f}(t, \underline{x})}{\partial x_i \; \partial x_j} \right]$ is negative semidefinite for $0 \leq t < \infty$ and $\| \underline{x} \| < \infty$ .

## Theorem 2 (Uniform Asymptotic Stability in the Large)

Let $\underline{f}(t, \underline{x})$ be continuous in $\underline{x}$ at $\underline{x} = \underline{0}$ for $0 \leq t < \infty$ ; let $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$ ; and let $\gamma (\underline{x}) \equiv \lim \sup_{t \to \infty} \underline{x}_T \; \underline{f}(t, \underline{x})$.

If $\underline{x}_T \; \underline{f}(t, \underline{x})$ is a strictly concave function of $\underline{x}$ for $0 \leq t < \infty$ and if either (I) $\gamma (\underline{x}) < 0$ for $\underline{x} \neq 0$, or (II) $\gamma (\underline{x})$ is strictly concave in $\underline{x}$, then $\underline{x} = \underline{0}$ is a __uniformly__, __asymptotically stable__ point in the large of the system given by equation (1).

## Notes About the Proof

The Liapunov function given by $V(t, \underline{x}) = \underline{x}_T \underline{x}$ is positive definite, tends to infinity with $\| \underline{x} \|$ , and has an infinitely small upper bound. The time derivative $\dot{V} = 2 \; \underline{x}_T \dot{\underline{x}} = 2 \; \underline{x}_T \underline{f}(t, \underline{x})$ is negative definite by Lemma 2. Thus, the conclusion follows by Massera's Theorem.

## Examples

The __equation__ $\dot{x} = - e^{-t} x$ is such that $x = 0$ is not an asymptotically stable point. From $\underline{x}_T \; \underline{f}(t, \underline{x})$ we have the following for $\gamma (x)$:

$$\gamma (x) = \lim \sup_{t \to \infty} \left\{ \underline{x}_T \underline{f}(t, \underline{x}) \right\} = \lim \sup_{t \to \infty} \left\{ - e^{-t} x^2 \right\} = 0.$$

Thus, $\gamma (x)$ does not satisfy either (I) or (II).

Another example is the system $\dot{x} = -(1 + e^{-t})x$ , for which $x = 0$ is an asymptotically stable point in the large, as shown by Theorem 2. In this case $\psi(x)$ is

$$\psi(x) = \lim_{t \to \infty} \sup \left\{ -(1 + e^{-t})x^2 \right\} = -x^2 ,$$

which satisfies both (I) and (II) in Theorem 2.

## Corollary 2

Let $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$ , let

$$\psi(\underline{x}) \equiv \lim_{t \to \infty} \sup \left\{ \underline{x}_T f(t, \underline{x}) \right\} ,$$

and let $\underline{f}(t, \underline{x})$ be a twice continuously differentiable function of $\underline{x}$ for $0 \leq t < \infty$ .

If the matrix $\left[ \dfrac{\partial^2 \underline{x}_T \underline{f}(t, \underline{x})}{\partial x_i \partial x_j} \right]$ is negative definite for $0 \leq t < \infty$ and $\|\underline{x}\| < \infty$ , and if either (I) $\psi(\underline{x}) < \infty$ for $\underline{x} \neq 0$ , or (II) $\psi(\underline{x})$ is twice continuously differentiable, and the matrix $\left[ \dfrac{\partial^2 \psi(\underline{x})}{\partial x_i \partial x_j} \right]$ is negative definite for $\|\underline{x}\| < \infty$ , then the point $\underline{x} = \underline{o}$ is a uniformly, asymptotically stable point in the large of the system (1).

## Theorem 3 (Instability)

Let $\underline{f}(t, \underline{x})$ be continuous in $\underline{x}$ at $\underline{x} = \underline{0}$ for $o \leq t < \infty$ let $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$ and let $\psi(\underline{x}) \equiv \lim_{t \to \infty} \inf \left\{ \underline{x}_T \underline{f}(t, \underline{x}) \right\}$ . If $\underline{x}_T \underline{f}(t, \underline{x})$ is a strictly convex function of $\underline{x}$ for $0 \leq t < \infty$ and if either (I) $\psi(\underline{x}) > 0$ for $\underline{x} \neq \underline{0}$, or (II) $\psi(\underline{x})$ is strictly convex in $\underline{x}$, then $\underline{x} = \underline{0}$ is an unstable equilibrium point of the system (1).

## Notes About the Proof

Consider the function $V(t, \underline{x}) = \underline{x}_T \underline{x}$, which is positive definite and has an infinitesimally small upper bound. By Lemma 3, $\dot{V} = 2 \underline{x}_T \underline{f}(t, \underline{x})$ is positive definite. The conclusion follows by Liapunov's Instability Theorem.

## Corollary 3

Let $\underline{f}(t, \underline{0}) = \underline{0}$ for $0 \leq t < \infty$, let $\gamma(\underline{x}) \equiv \lim_{t \to \infty} \inf \left\{ \underline{x}_T \underline{f}(t, \underline{x}) \right\}$, and let $\underline{f}(t, \underline{x})$ be a twice continuously differentiable function of $\underline{x}$ for $0 \leq t < \infty$. If the matrix $\left[ \dfrac{\partial^2 \, \underline{x}_T \, \underline{f}(t, \underline{x})}{\partial x_i \, \partial x_j} \right]$ is positive definite for $0 \leq t < \infty$ and $\| \underline{x} \| < \infty$, and if either (I) $\gamma(\underline{x}) > 0$ for $\underline{x} \neq \underline{0}$, or (II) $\gamma(\underline{x})$ is twice continuously differentiable, and the matrix $\left[ \dfrac{\partial^2 \gamma}{\partial x_i \, \partial x_j} \right]$ is positive definite for $0 \leq t < \infty$, then $\underline{x} = \underline{0}$ is an unstable equilibrium point of (1).

## Mangasarian's Theorem 2 V.S. Krasovskii's Theorem

Theorem 2 differs with Krasovskii's work because of the different choices for the Liapunov functions in the two theorems. The differences can be summarized in the following way:

(a) Krasovskii's theorem is for autonomous cases while Theorem 2 is for non-autonomous systems.

(b) Krasovskii requires that $\underline{f}$ be differentiable while Theorem 2 requires that $\underline{f}$ be only continuous at $\underline{x} = \underline{0}$.

Examples of Mangasarian's work are given in the compendium at the end of this section.

## CHANG'S WORK

In reference [9], Chang discusses a kinetic Liapunov function. This is a Liapunov function of the first derivatives of the state variables. Therefore, this idea is not new but is an extension of the Liapunov function used in the proof of Krasovskii's theorem in which $\| \underline{\dot{x}} \|^2$ is used as the function. In general, the technique consists of finding the sufficient conditions for the time derivative $\dot{x}$ to approach zero. As $\dot{x}$ approaches zero,

the system arrives at one of its equilibrium points $\underline{x}_e$. This method has the following advantages:

(1)  For different sets of steady-state inputs, the equilibrium points $\underline{x}_e$

vary.  In cases where the kinetic Liapunov function is independent of

$\underline{x}_e$, the stability problem can be settled once and for all.

(2)  The kinetic Liapunov function leads to linearization.

One main disadvantage is the severity of the resulting sufficient conditions which are placed on the system by the kinetic Liapunov function.

Chang's stability analysis is applied to a continuous time dynamical

system defined by

$$\dot{\underline{x}} = f\,(\underline{x},\ t)\ ,\tag{1}$$

where $\underline{f}$ is a continuous function of $\underline{x}$ and $t$, having continuous finite first

partial derivatives of $\underline{x}$.

The definition of the kinetic Liapunov function $K(\underline{x},\ \dot{\underline{x}},\ t)$ is as follows:

(i)  $K(\underline{x},\ \dot{\underline{x}},\ t)$ is continuous in $\underline{x}$, $\dot{\underline{x}}$, and $t$, positive definite in $\dot{\underline{x}}$ and

is finite for finite $\|\dot{\underline{x}}\|$ ; that is

$$0 < \alpha(\ \|\dot{\underline{x}}\|\ ) \leq K\,(\underline{x},\ \dot{\underline{x}},\ t) \leq \beta\,(\ \|\dot{\underline{x}}\|\ ),\tag{2}$$

where $\alpha$ and $\beta$ are nondecreasing functions of $\|\dot{\underline{x}}\|$ ,

$\alpha\,(0) = \beta\,(0) = 0$, and $\alpha\,(\ \|\dot{\underline{x}}\|\ )$ and

$\beta\,(\ \|\dot{\underline{x}}\|\ )$ approach infinity as $\|\dot{\underline{x}}\|$ approaches infinity.

(ii)  The total time derivative $dK/dt$ satisfies,

$$\frac{d\,K}{d\,t} < -\,\gamma\,(\ \|\dot{\underline{x}}\|\ ) < 0\ ,\tag{3}$$

except at $\dot{\underline{x}} = 0$, where $\gamma\,(0) = 0$. The inequalities (2) and (3) are to be

satisfied for all values of $\underline{x}$.

## Main Theorem

"A dynamical system always converges to one of its equilibrium points $\underline{x}_e$

if (i) there exists a kinetic Liapunov function, and (ii) for any given r > 0,

there exists an m(r) > 0, such that $\| \underline{x} - \underline{x}_e \| \geqslant$ r for all $\underline{x}_e$ implies that

$\| \underline{\dot{x}} \| > m(r)$."

In the following proposition, Chang shows why the existence of the kinetic

Liapunov function is a rather stringent condition.

**Proposition 1.** "A dynamical system having one or more permanent unstable

equilibrium points cannot have a kinetic Liapunov function."

This means that kinetic Liapunov functions do not exist for stable

equilibrium points of the system if that system has a permanent unstable

equilibrium point; that is, a point where $\underline{f}(\underline{x}_e, t) = 0$ for all t and where

the system is unstable. The reason for this is that kinetic Liapunov functions

must exist for all $\underline{x}$, but in the neighborhoods of unstable points they do not

exist.

**Proposition 2.** "For time-invariant systems condition (ii) of the kinetic

Liapunov function can be relaxed by allowing $\left\{ \dfrac{dK(\underline{x}, \underline{\dot{x}})}{dt} \right\} = 0$

at a finite number of isolated points of the state space."

The results of the main theorem and proposition 1 lead to the following

corollary:

**Corollary 1.** "A dynamical system is uniformly asymptotically stable in the

large if there exists a kinetic Liapunov function and only one stable equilib-

rium point."

The next corollary gives one way of finding a kinetic Liapunov function

for a time-invariant system. The proof of this corollary will be presented since

it is an example of a kinetic Liapunov function.

**Corollary 2.** "A time-invariant dynamical system having one stable equilibrium

point is asymptotically stable in the large if there exists a positive definite

symmetrical constant matrix $\underline{B}$ such that

$$\underline{A}_T (\underline{x}) \; \underline{B} \; + \underline{B} \; \underline{A}_T (\underline{x}), \tag{4}$$

is also positive definite for every $\underline{x}$, where

$$\underline{A}(\underline{x}) \equiv - \begin{bmatrix} \dfrac{\partial f_i}{\partial x_j} \end{bmatrix} , \quad \dot{x}_i = f_i (\underline{x}). \text{"} \tag{5}$$

Proof

Let the candidate for a kinetic Liapunov function be $\dot{\underline{x}}_T \underline{B} \; \dot{\underline{x}}$. From equations (1) and (5) we have $\ddot{\underline{x}} = - \underline{A} \; \dot{\underline{x}}$. Thus, the time derivative of $K = \dot{\underline{x}}_T \underline{B} \; \dot{\underline{x}}$ is

$$\dot{K} = \ddot{\underline{x}}_T \underline{B} \; \dot{\underline{x}} + \dot{\underline{x}}_T \underline{B} \; \ddot{\underline{x}} + \dot{\underline{x}}_T \dot{\underline{B}} \; \dot{\underline{x}}$$

$$= - \dot{\underline{x}}_T \left\{ \underline{A}_T \underline{B} \; + \underline{B} \; \underline{A} - \dot{\underline{B}} \right\} \dot{\underline{x}} .$$

Both $\underline{B}$ and $\underline{A}_T \underline{B} + \underline{B} \; \underline{A} - \dot{\underline{B}}$ must be positive definite for $\dot{\underline{x}}_T \underline{B} \; \dot{\underline{x}}$ to be a kinetic Liapunov function. In case $\underline{B}$ is a constant, then $\dot{\underline{B}} = \underline{0}$, and the corollary is proved.

It should be pointed out that for uniformly asymptotic stability in the large the expression $\underline{A}_T \underline{B} + \underline{B} \; \underline{A}$ must be positive definite for the same constant $\underline{B}$ and for all possible $\underline{A}$'s which are generated as $\underline{x}$ varies.

Chang uses the following algebraic relations in his examples, which are in the compendium at the end of this section.

(1) For arbitrary $\underline{A}$ there exists a symmetrical, positive definite $\underline{B}$ such that $\underline{A}_T \underline{B} + \underline{B} \; \underline{A}$ is positive definite, if and only if all the eigenvalues of $\underline{A}$ are positive. The Matrix $\underline{B}$ is called an orientation of $\underline{A}$.

(2) Let $\underline{A}(\underline{x}) = \underline{A}_o + f(\underline{x}) \underline{A}_1$ , where $\underline{A}_o$ and $\underline{A}_1$ are constant matrices, and $L \leqslant f(\underline{x}) \leqslant U$. A symmetrical, positive definite matrix $\underline{B}$ is an orientation of $\underline{A}(\underline{x})$ for all $\underline{x}$, if and only if $\underline{B}$ is a common orientation of both $\underline{A}_o + L \underline{A}_1$ and $\underline{A} + U \underline{A}_1$ .

(3) There exists a common orientation $\underline{B}$ for matrices $\underline{A}$ and $\underline{C}$ if and only if there is a nonsingular transformation $\underline{T}$ such that both $\underline{T}\ \underline{A}\ \underline{T}^{-1}$ and $\underline{T}\ \underline{C}\ \underline{T}^{-1}$ are positive definite. Then $\underline{B} = \underline{T}_T\ \underline{T}$ .

## INGWERSON'S WORK

Ingwerson published the main result of his thesis investigations [14] , in reference [10] , in 1961. This result was a method of generating Liapunov functions by an integration technique. In conjunction with reference [10] is an intense discussion between Vogt and Ingwerson, in references [11] and [12] , regarding the preciseness of certain definitions and theorems. In reference [13] , Rodden shows that Ingwerson's method of generating Liapunov functions is amenable to computational machine procedures.

Summarizing, Ingwerson's method consists of solving a matrix equation in closed form, modifying the result, and performing a type of double integration. This method is well motivated as we will see in the following paragraphs; can often be carried out analytically, and often gives excellent results. This method of construction imposes only continuity and uniqueness conditions on the right-hand sides of the systems' differential equations. In principle, Liapunov functions of piecewise linear systems can be constructed with this method. Under certain conditions, discontinuous systems can be handled. In the next paragraphs we briefly present Ingwerson's historical and theoretical motivation for the development of his method. This material comes from reference [10] .

Liapunov considered the nonlinear autonomous system, $\dot{\underline{x}} = \underline{f}(\underline{x})$, where $\dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{\partial x_i}$ . That is, the curl of vector $\underline{f}$ vanishes identically. Because of this, the following line integral is uniquely integrable:

$$V(\underline{x}) = - \int_{\underline{0}}^{\underline{x}} \underline{f}\,(\underline{x}) \cdot d\,\underline{x} \ .$$

The time derivative of $V(\underline{x})$ is $\{-\underline{f}_T\underline{f}\}$, which is nonpositive. If V is positive definite, then the system is stable. We have asymptotic stability in the large if $V \rightarrow \infty$ as $\|\underline{x}\| \longrightarrow \infty$. A condition for this is that the eigenvalues of the matrix with elements $\dfrac{\partial^2 V}{\partial x_i \, \partial x_j}$ or $\dfrac{\partial fi}{\partial x_j}$ be positive.

Liapunov's example given above is a special case of <u>Krasovskii's Theorem</u> [15]. The system considered by Krasovskii is, $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, where $\underline{f}(\underline{0}, t) = \underline{0}$. The matrix $\underline{B}(\underline{x}, t)$ is made up of the elements $\dfrac{\partial fi}{\partial x_j}$. The theorem states that the system is asymptotically stable in the large if the eigenvalues of the symmetric part of $\underline{B} = \frac{1}{2}(\underline{B}_T + \underline{B})$ are negative. For the autonomous case. Krasovskii, in reference [2], used a Liapunov function of the form $V = \underline{f}_T\underline{f}$. In this more general case, [15], the positive definite form, $V = \underline{x}_T\underline{x}$, is used. The derivative of this V is $\dot{V} = \underline{x}_T\underline{f} + \underline{f}_T\underline{x}$.

To show that $\dot{V}$ is negative definite the scalar product of $\nabla\dot{V} \cdot \underline{x}$ is formed:

$$\nabla\dot{V} \cdot \underline{x} = \dot{V} + \underline{x}_T(\underline{B}_T + \underline{B})\underline{x}.$$

Rewriting this equation gives

$$\nabla\left\{\frac{\dot{V}}{(\underline{x}_T\underline{x})^{\frac{1}{2}}}\right\} \cdot \underline{x} = \underline{x}_T(\underline{B}_T + \underline{B})\underline{x} \Big/ \sqrt{\underline{x}_T\underline{x}} < 0,$$

by hypothesis. Thus, $\dot{V}$ is negative definite.

<u>Liapunov's</u> results for <u>linear</u>, <u>autonomous differential equations</u> are both necessary and sufficient. The system of equations he considered is

$$\dot{\underline{x}} = \underline{B}\,\underline{x}, \tag{1}$$

where $\underline{B}$ is an $n \times n$ constant matrix. The choice for a Liapunov function is

$$V = \underline{x}_T\,\underline{A}\,\underline{x}, \tag{2}$$

where $\underline{A}$ is another $n \times n$ constant matrix. The corresponding time derivative of V is

$$\dot{V} = \underline{x}_T (\underline{B}_T \underline{A} + \underline{A} \underline{B}) \underline{x} = - \underline{x}_T \underline{C} \underline{x} , \qquad (3)$$

where for asymptotic stability matrix $\underline{C}$ must be a positive definite, symmetric matrix. Thus $\underline{x} = \underline{0}$ is asymptotically stable if and only if the matrix $\underline{A}$ which satisfies

$$\underline{B}_T \underline{A} + \underline{A} \underline{B} = - \underline{C} \qquad (4)$$

for the above $\underline{C}$ has positive eigenvalues. This condition may be relaxed by allowing V to be negative semidefinite rather than negative definite. Then only one of the eigenvalues of $\underline{C}$ need be different from zero. A sufficient condition for the eigenvalues of $\underline{A}$ to be positive or zero is that those of B have negative real parts.

Before presenting Ingwerson's table of $\underline{A}$ and $\underline{C}$ matrices, we will consider a second order example of the above discussion. Define a second order linear system by the following set of equations:

$$\dot{x}_1 = x_2 ,$$

$$\dot{x}_2 = - a_2 x_1 - a_1 x_2 .$$

Let matrix $\underline{C}$ take the form

$$\underline{C} = \frac{1}{2a_1 a_2} \begin{bmatrix} 0 & 0 \\ 0 & 2a_1 \end{bmatrix} ;$$

thus, matrix $\underline{A}$ is given by

$$\underline{A} = \frac{1}{2a_1 a_2} \begin{bmatrix} a_2 & 0 \\ 0 & 1 \end{bmatrix} .$$

The corresponding Liapunov function and its derivative are

$$V = \underline{x}_T \underline{A}\ \underline{x} = \frac{1}{2a_1\ a_2}\ (a_2\ x_1^2 + x_2^2)\ ,$$

$$V = -\ \underline{x}_T \underline{C}\ \underline{x} = -\ \frac{x_2^2}{a_2}.$$

For the general linear, autonomous case the matrix $\underline{B}$ is given by

$$\underline{B} =
\begin{vmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1
\end{vmatrix}
\qquad (5)$$

For systems up to fourth order, Ingwerson [10] , gives a table for the various $\underline{A}$'s and $\underline{C}$'s corresponding to equations (4) and (5). This table will now be reproduced; in each matrix $\underline{A}$ and $\underline{C}$ in the table there should appear, an extra scalar multiplier similar to the one given above in the second order case.

## Table of Matrices $\underline{A}$ and $\underline{C}$

2nd Order

$$A_1 =
\begin{vmatrix}
a_2 & 0 \\
0 & 1
\end{vmatrix}
\qquad
A_2 =
\begin{vmatrix}
a_1^2 + a_2 & a_1 \\
a_1 & 1
\end{vmatrix}$$

$$C_1 =
\begin{vmatrix}
0 & 0 \\
0 & 2a
\end{vmatrix}
\qquad
C_2 =
\begin{vmatrix}
2a_1 a_2 & 0 \\
0 & 0
\end{vmatrix}$$

## 3rd Order

$$A_1 = \begin{bmatrix} a_3^2 & a_2 a_3 & 0 \\ a_2 a_3 & a_1 a_3 + a_2^2 & a_3 \\ 0 & a_3 & a_2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} a_1 a_3 & a_3 & 0 \\ a_3 & a_1^2 + a_2 & a_1 \\ 0 & a_1 & 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2(a_1 a_2 - a_3) \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(a_1 a_2 - a_3) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a_1 a_2^2 - a_2 a_3 + a_1^2 a_3 & a_1^2 a_2 & a_1 a_2 - a_3 \\ a_1^2 a_2 & a_1^3 + a_3 & a_1^2 \\ a_1 a_2 - a_3 & a_1^2 & a_1 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 2a_3 (a_1 a_2 - a_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## 4th Order

$$A_1 = \begin{bmatrix} a_3^2 a_4 & a_4 a_3^2 & a_2 a_3 a_4 - a_1 a_4^2 & 0 \\ a_4 a_3^2 & a_1 a_4^2 + a_3^2 & a_2 a_3^2 & a_3 a_4 \\ a_2 a_3 a_4 - a_1 a_4^2 & a_2 a_3^2 & a_1 a_3^2 + a_2^2 a_3 - a_1 a_2 a_4 - a_3 a_4 & a_3^2 \\ 0 & a_3 a_4 & a_3^2 & a_2 a_3 - a_1 a_4 \end{bmatrix}$$

$$R_2 = a_1 a_2 - a_3 \; ; \quad R_3 = a_1 a_2 a_3 - a_3^2 - a_1^2 a_4,$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2R_3 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2R_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{array}{|c|c|c|c|}
\hline
a_1a_4^2 & a_1a_3a_4 & a_3a_4 & 0 \\
\hline
a_1a_3a_4 & a_1^2a_3 + a_4R_2 & a_3^2 + a_1^2a_4 & a_1a_4 \\
\hline
a_3a_4 & a_3^2 + a_1^2a_4 & a_1^2a_3 + a_2a_3 - a_1a_4 & a_1a_3 \\
\hline
0 & a_1a_4 & a_1a_3 & a_3 \\
\hline
\end{array}$$

$$A_3 = \begin{array}{|c|c|c|c|}
\hline
a_4R_2 & a_1^2a_4 & a_1a_4 & 0 \\
\hline
a_1^2a_4 & a_2R_2 - a_1a_4 + a_1^2a_3 & a_1^2 + a_2 & R_2 \\
\hline
a_1a_4 & a_1^2a_2 & a_1^3 + a_3 & a_1^2 \\
\hline
0 & R_2 & a_1^2 & a_1 \\
\hline
\end{array}$$

$$C_3 = \begin{array}{|c|c|c|c|}
\hline
0 & 0 & 0 & 0 \\
\hline
0 & 2R_3 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad
C_4 = \begin{array}{|c|c|c|c|}
\hline
2a_4R_3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
\hline
\end{array}$$

$$A_4 = \begin{array}{|c|c|c|c|}
\hline
a_3R_3 + a_4(a_2R_2 - a_1a_4) & a_3(a_2R_2 - a_1a_4) & a_1R_3 + a_4R_2 & R_3 \\
\hline
a_3(a_2R_2 - a_1a_4) & (a_2^2 - a_4)R_2 + a_1a_4(a_1^2 - a_2) & a_1a_2R_2 & a_2R_2 - a_1a_4 \\
\hline
a_1R_3 + a_4R_2 & a_1a_2R_2 & a_1^2R_2 + a_1a_4 & a_1R_2 \\
\hline
R_3 & a_2R_2 - a_1a_4 & a_1R_2 & R_2 \\
\hline
\end{array}$$

Ingwerson wanted a _method of generating Liapunov functions_ which would

solve the linear stability problem exactly and which would give sufficient

conditions for the stability of nonlinear systems under small disturbances;

with the hope, that the conditions for stability under large deviations would

also be useful in applications, although the results would be very conserva-

tive. His method is based on the observation that the quadratic part of the

Liapunov function must give stability information for the linearized equations of motion of the system. Ingwerson also based some of the development given below on the example of Liapunov, given above, and on Krasovskii's Theorem and proof. The outline given below is based on the material presented in references [3] and [10] .

The nonlinear autonomous system

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

is differentiated with respect to time to give

$$\ddot{\underline{x}} = \underline{B}(\underline{x}) \, \dot{\underline{x}} \; ,$$

where $\underline{B}(\underline{x})$ is the Jacobian matrix of $\underline{f}(\underline{x})$. That is, the elements of $\underline{B}$ are $\partial f_i / \partial x_j$ . The matrix equation

$$\underline{B}_T \underline{A} + \underline{A} \, \underline{B} = - \underline{C}$$

may be solved for a matrix $\underline{A}(\underline{x})$ for a given choice of matrix $\underline{C}$, as in the linear results of Liapunov. In the linear case $\underline{A}$ is a matrix with elements of the form $\dfrac{1}{2} \dfrac{\partial^2 V}{\partial x_i \, \partial x_j}$ . Ingwerson observed that if this were also true for matrix $\underline{A}(\underline{x})$, then a Liapunov function could be found by performing two integrations. His reasoning being that for any procedure based on this idea, the correct answers are obtained for the linear problem, as is desired in any general method.

There are necessary conditions which must be fulfilled by the elements of $\underline{A}(\underline{x})$ if these elements are to be second partial derivatives of a scalar function. First, the elements must be such that the "mixed partials" of V are equal, assuming sufficient continuity restrictions on the partial derivatives; this implies that $\underline{A}(\underline{x})$ must be symmetric, which is the case.

Second, the elements of $\underline{A}(\underline{x})$ must be such that

$$\frac{\partial a_{ij}}{\partial x_k} = \frac{\partial a_{ik}}{\partial x_j} ,$$

again assuming sufficient continuity requirements. This condition is usually

not fulfilled in the matrices, $\underline{A}(\underline{x})$, derived above. But the $\underline{A}(\underline{x})$ obtained

does have the desirable feature that the Liapunov function derived from

$A(\underline{0})$ is valid in the vicinity of the origin. This Liapunov function gives

stability information about the linear approximation of $\underline{\dot{x}} = \underline{f}(\underline{x})$ in the

vicinity of $\underline{x} = \underline{0}$.

From the above "impasse" Ingwerson salvaged the following procedure.

He found, as one can see in the examples at the end of this section, that

good results could be obtained if the matrix $\underline{A}(\underline{x})$ is obtained as above and

in each element $a_{ij}$ (i $\neq$ j) of $\underline{A}(\underline{x})$ only the variables $x_i$ and $x_j$ are retained,

the other variables are replaced by zero. Thus, $a_{ij}(\underline{x}) = a_{ij}(x_i, x_j)$. There-

fore, the gradient of the scalar function V is found from the matrix integra-

tion, which is an "elementwise" indefinite integration procedure, of

$\underline{A}(\underline{x}) = \underline{A}(x_i, x_j)$,

$$\underline{\nabla} V = \int_{\underline{0}}^{\underline{x}} \underline{A} (x_i, x_j) \, d\underline{x} .$$

The resulting vector $\underline{\nabla} \underline{V}$ has curl zero and, hence, is the gradient of a

scalar function V. The line integration of $\underline{\nabla}\underline{V}$ gives the scalar V. Then

V and $\dot{V} = \underline{\nabla}\underline{V}_T \underline{f} (\underline{x})$ are checked for the properties of definiteness

or semidefiniteness as the case may be.

The table given by Ingwerson of the $\underline{A}$ and $\underline{C}$ matrices for linear systems

can be used for this nonlinear problem. From the table and the corresponding

$\underline{B}(x)$ matrix, a matrix $\underline{A}$ is chosen, except the constant elements in $\underline{A}$ are

replaced by the corresponding variable elements obtained from the $\underline{B}$ - matrix. But one should realize that there is much ingenuity required in the choice of matrix $\underline{A}$ and in deciding when it is feasible to choose a combination of the A's, and thus a linear combination of corresponding "Liapunov functions", and in determining when some of the elements in $\underline{A}(\underline{x})$ should be modified.

In summary, the Ingwerson method is, [3]:

(1) Obtain the matrix $\underline{B}(\underline{x})$ from the given system, $\underline{\dot{x}} = \underline{f}(\underline{x})$; and choose a symmetric $\underline{C}$. Then "calculate" matrix $\underline{A}(\underline{x})$ from $\underline{B}_T \underline{A} + \underline{A} \ \underline{B} = - \ \underline{C}$.

(2) Cross out the terms in $\underline{A}(\underline{x})$ which violate $a_{ij} = a_{ji}$; and cross out those terms in the element $a_{ij}$ which violate $a_{ij} = a_{ij} \ (x_i, x_j)$.

(3) Perform the integrations,

$$\underline{\nabla V} = \int_{\underline{o}}^{\underline{x}} \underline{A} \ (\underline{x}) \ d \ \underline{x} \ ,$$

and

$$V = \int_{\underline{o}}^{\underline{x}} \underline{\nabla V} \cdot d \ \underline{x} \ .$$

(4) Find $\dot{V}$ from the equation

$$\dot{V} = \underline{\nabla V}_T \ \underline{f} \ (\underline{x}) \ .$$

(5) Test $V$ and $\dot{V}$ for the appropriate definiteness properties.

(In step 1, Ingwerson's table of "linear A-matrices" provides a useful first approximation for the desired $\underline{A}(\underline{x})$'s if the constants $a_{ij}$ are replaced by the corresponding variables $a_{ij} \ (x_i, x_j)$.

## SZEGO'S WORK

In reference [16] Szego developed a procedure similar to the other techniques presented in this section. The systems whose stability is to

be investigated by this technique are nonlinear autonomous systems with

nonlinearities representable in polynominal form. The V-functions used

in the analysis are derived from the particular system under investigation

by a procedure based upon a class of functions that Szegö calls "the class

of generating V-functions." Such a generating V-function has the form

$V(\underline{x}) = \underline{x}_T \underline{A}(\underline{x}) \; \underline{x}$, where $\underline{A}(\underline{x}) = \begin{bmatrix} a_{ij} \; (x_i, \; x_j) \end{bmatrix}$ , $a_{ij} = a_{ji}$ . The

coefficients $a_{ij}$ are determined such that $\overset{\bullet}{V}$ possesses the desired properties

specified in Szegö's Stability Theorems.

The system under investigation is

$$\overset{\bullet}{\underline{x}} = \underline{f}(\underline{x}), \tag{1}$$

which is assumed to have only one equilibrium point, $\underline{x} = \underline{0}$. The nonlinearity,

$\underline{f}(\underline{x})$, is representable in polynominal form. Szego's theorem on <u>asymptotic</u>

<u>stability in the large</u> is presented in the usual form. But for <u>local</u>

<u>stability</u> his theorem is stated in the following way:

"A sufficient condition that the solution $\underline{x} = \underline{0}$ of the system (1) is

locally stable is that there exists a positive definite scalar function

$V = \phi (\underline{x})$ with continuous first partial derivatives, such that its total

time derivative with respect to the system (1) has the form

$$\overset{\bullet}{V} = \Theta(\underline{x}) \; g \left[ \xi \; (\underline{x}) \right] , \tag{2}$$

where $\Theta (\underline{x})$ is a semidefinite function not identically equal to zero on any

nontrivial solution of (1), and $g(\underline{x})$ is indefinite on a closed surface. In

particular assume that $\xi (\underline{x}) = 0$ is a closed surface, or a family of

closed surfaces, and $g$ (u) is such that g (0) = 0 and

$g(u)/u > 0$ ."

In this theorem the meaning of being <u>indefinite on a closed surface</u>

is: "A function defined by $\phi = \phi (\underline{x})$ is indefinite on a closed

surface if $\phi(\underline{x}) = 0$ is a closed bounded surface and if the sign of $\phi(\underline{x})$ inside $\phi(\underline{x}) = 0$ is different from the sign of $\phi(\underline{x})$ outside $\phi(\underline{x}) = 0$." This is the essence of the difference between Szegö's Method and some of the other methods; that is, $\dot{V}$ is allowed to be indefinite on a closed surface. This then leads to the possibility of approximating the limit surface of system (1), if it exists.

With reference to this limit surface, we now briefly describe the construction procedure used to obtain an approximation of this surface. We denote by $V = \mathcal{V}c$ that surface of the family $V = \phi(\underline{x})$ which is circumscribed to the surface $\xi(\underline{x}) = 0$, and $V = \mathcal{V}m$ the inscribed surface. Then for every surface of the family $V = \phi(\underline{x})$ such that $V = Vc \geqslant \mathcal{V}c$ , $\dot{V}_c$ will be semidefinite of a particular sign; and for every surface with $Vc \leqslant \mathcal{V}m$, $\dot{V}_c$ will be semidefinite of opposite sign. The closed bounded surface, $\xi(\underline{x}) = 0$, lies between the surfaces $\phi(\underline{x}) = \mathcal{V}c$ and $\phi(\underline{x}) = \mathcal{V}m$. If $\mathcal{V}c = \mathcal{V}m$, then $\xi(\underline{x}) = 0$ identifies exactly the boundary between stability and instability. This boundary is a limit set of the system. If this constitutes the only limit cycle for the system, then it will be stable and the solutions will be bounded in the neighborhood of $\underline{x} = \underline{0}$ for $\dot{V}c \leqslant 0$. The limit cycle will be unstable and some solutions will be unbounded if $\dot{V}_c \geqslant 0$. From the above discussion one is given a method of approximating limit cycles, as will be demonstrated in the examples.

Szego's main aim is to investigate the stability of the solution $\underline{x} = \underline{0}$ so as to identify analytically the regions of stability or instability of the solution, $\underline{x} = \underline{0}$, and the boundaries between them by using the Second Method of Liapunov. The "generating V-function is a polynomial in $x_1, \ldots, x_n$ represented as follows:

$$V(\underline{x}) = \underline{x}_T \underline{A}(\underline{x}) \underline{x} , \qquad (3)$$

where the elements of $\underline{A}$ are of the form $a_{ij} = a_{ij} (x_i, x_j),$ and are not dependent upon $x_n$. The reason for the last restriction is that usually limit cycles have at most two real intersections with the hyperplanes $x_i$ = constant, i = 1, 2, ... , n-1. The unknown parameters $a_{ij}$ are determined such that $\overset{\bullet}{V}$ possesses the desired properties. Since $\underline{f}(\underline{x})$ in (1) is representable in polynominal form, then the coefficients $a_{ij}$, up to a constant factor, satisfy simple algebraic equations.

From Szegö's formulation of the method of generating V-functions, the problem of the stability of system (1) is reduced to the investigation of the properties of the solution of a matrix equation and the values of certain constants. Let us consider system (1) in the form:

$$\overset{\bullet}{\underline{x}} = \underline{B} (\underline{x}) \underline{x} , \tag{4}$$

where $\underline{B}$ is not generally known uniquely. The V-function is given by equation (3) where the matrix $\underline{A}$ is symmetric and the $a_{ij}$ elements are of the form $a_{ij} = a_{ij} (x_i, x_j)$. From equations (3) and (4) we get the time derivative of V:

$$\overset{\bullet}{V} = \underline{x}_T \left\{ \underline{B}_T \underline{A} + \underline{A} \underline{B} + \overset{\bullet}{A} \right\} \underline{x} . \tag{5}$$

As pointed out in Szegö's Theorem, we want the form of V to be

$$\overset{\bullet}{V} = \ominus (\underline{x}) \left\{ \underline{x}_T \underline{c} (\underline{x}) \underline{x} - K \right\} \tag{6}$$

where $\phi = \underline{x}_T \underline{c}(\underline{x})\underline{x}$ is a definite scalar function such that $\phi \leq K$, $\ominus (\underline{x})$ is a semidefinite scaler function not equal to zero on a trajectory of the system (1), and K is a nonnegative constant. If we assume that $\ominus (\underline{x}) = \underline{x}_T \underline{\Theta} \underline{x}$ , then $\overset{\bullet}{V}$ becomes

$$\overset{\bullet}{V} = \underline{x}_T \left\{ \underline{\Theta} (\underline{x}) \underline{x} \underline{x}_T \underline{c} (\underline{x}) - K \underline{\Theta} (\underline{x}) \right\} \underline{x} . \tag{7}$$

Thus, the problem of stability is reduced to the solution of the following

matrix equation for matrix $\underline{A}$:

$$\underline{x}_T \left\{ \underline{B}_T \; (\underline{x}) \; \underline{A} \; (x_i, x_j) \; + \; \underline{A} \; (x_i, x_j) \; \underline{B} \; (\underline{x}) \right\} \; \underline{x} \; =$$

$$= \underline{x}_T \left\{ \underline{\Theta} (\underline{x}) \; \underline{x} \; \underline{x}_T \; \underline{c} \; (\underline{x}) \; - \; K \; \underline{\Theta} \; (\underline{x}) \right\} \; \underline{x} \; , \tag{8}$$

which is derived from equations (5) and (7) and from the particular form

of $\underline{A}$ which is required for system (1). For second and fourth order systems,

the solution of (8) is satisfactory for stability analysis.

For third order systems, Szego suggests the following V-function:

$$V = \underline{x}_T \left\{ \underline{D} \; \underline{x}_T \underline{x} \; \underline{E} \; + \; \underline{A} \right\} \; \underline{x} \; , \tag{9}$$

where $\underline{A}$, $\underline{D}$, and $\underline{E}$ are symmetric matrices of the form, $\underline{A} = \left[ a_{ij} \; (x_i, \; x_j) \right]$ ,

$\underline{B} = \left[ b_{ij} \; (x_i, \; x_j) \right]$ and $\underline{E} = \left[ e_{ij} \; (x_i, \; x_j) \right]$ . From equation (7) and

and the total time derivative of (9), the problem of stability for the third -

order system is reduced to the computation of the 18 elements of the matrices $\underline{A}$,

$\underline{D}$ and $\underline{E}$ which satisfy the equation:

$$\underline{x}_T \left\{ (\underline{B}_T \underline{D} \; + \; \underline{D} \; \underline{B}) \; \underline{x} \; \underline{x}_T \underline{E} \; + \underline{D} \; \underline{x} \; \underline{x}_T (\underline{B}_T \underline{E} \; + \; \underline{E} \; \underline{B}) \; + \underline{B}_T \underline{A} \; + \; \underline{A} \; \underline{B} \right\} \underline{x} \; =$$

$$= \underline{x}_T \left\{ \underline{\Theta} \; (\underline{x}) \; \underline{x} \; \underline{x}_T \underline{c} \; (\underline{x}) \; - \; K \; \underline{\Theta} (\underline{x}) \right\} \underline{x}. \tag{10}$$

For a fifth - order system one can use the following V-function:

$$V = \underline{x}_T \left\{ \underline{F} \; \underline{x} \; \underline{x}_T \underline{G} \; \underline{x} \; \underline{x}_T \underline{H} \; + \; \underline{D} \; \underline{x} \; \underline{x}_T \underline{E} \; + \; \underline{A} \right\} \; \underline{x} \; , \tag{11}$$

where $\underline{A}$, $\underline{F}$, $\underline{G}$ and $\underline{H}$ are symmetric matrices and either $\underline{D}$ or $\underline{E}$ is

symmetrical, and

$$\underline{A} = \left[ a_{ij} \; (x_i, \; x_j) \right] \; , \qquad \underline{D} = \left[ d_{ij}(x_i, \; x_j) \right] \; , \qquad \underline{E} = \left[ e_{ij}(x_i, \; x_j) \right] \; ,$$

$$\underline{H} = \left[ h_{ij}(x_i, \; x_j) \right] \; , \qquad \underline{G} = \left[ g_{ij}(x_i, \; x_j) \right] \; , \qquad \underline{F} = \left[ f_{ij}(x_i, \; x_j) \right] \; .$$

These six matrices must satisfy a matrix equation similar to equations (8)

and (10).

In summary, we give a list of several observations which can be made about the procedure.

(1) In solving the matrix equations in (8) and (10) several simplifying assumptions are usually made and the final result is a modification of the original proposed V-function. Examples of this procedure are given at the end of the section.

(2) Similar V-functions can be constructed for every finite-order autonomous system.

(3) This method of generating Liapunov functions uses the "geometric meaning" of the functions and therefore gives a method to estimate limit surfaces, if they exist.

(4) This same method for constructing Liapunov functions can also use the approach of Krasovskii by introducing a V-function of the form

$$V = \underline{f}_T (\underline{x}) \; \underline{A} \; (x_i, x_j) \; \underline{f} \; (\underline{x}) \; , \tag{12}$$

whose total time derivative contains the Jacobian matrix of the system (1) in place of the $\underline{B}(\underline{x})$ matrix. The advantage is that the Jacobian Matrix is uniquely defined by the system while the $\underline{B}(\underline{x})$ matrix is not uniquely defined.

(5) The disadvantage of this method is the "usually story", too much computation is required for most systems.

## COMPENDIUM OF EXAMPLES

Example 1, [4]                    Linear Circuit

We apply Krasovskii's theorem to the following linear circuit

$$\dot{e}_1 = - 4Ke_1 + 4Ke_2,$$

$$\dot{e}_2 = 2Ke_1 - 6Ke_2,$$

where the Jacobian matrix is

$$\underline{F}(\underline{e}) =$$

| - 4K | 4K |
|------|-----|
| 2K | -6K |

.

Thus, the $\hat{\underline{F}}$ matrix becomes

$$\hat{\underline{F}} = \underline{F} + \underline{F}_T =$$

| -8K | 6K |
|------|-------|
| 6K | -12K |

,

where the eigenvalues of $\hat{\underline{F}}$ are negative if $K > 0$. Therefore the circuit is asymptotically stable everywhere.

Example 2, [5]          Second Order, Nonlinear System

We have a nonlinear system given by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - Kf(x_1) - wx_2$$

The candidate for a Liapunov function in this case is chosen to be of the form

$$V = \underline{f}_T \underline{A} \underline{f} = a_{11} f_1^2 + 2a_{12} f_1 f_2 + a_{22} f_2^2 ,$$

where $f_1 = x_2$           and   $f_2 = - Kf(x_1) - wx_2$ . For

a positive definite V we must require that $f(x_1) = 0$ implies that $x_1 = 0$,

$a_{11} > 0$, $a_{11} a_{22} - a_{12}^2 > 0$. The corresponding $\dot{V}$ is

$$\dot{V} = \left\{ \frac{\partial V}{\partial f_1} \frac{df_1}{dt} + \frac{\partial V}{\partial f_2} \frac{df_2}{dt} \right\}$$

$$= \left\{ \frac{\partial V}{\partial f_1} \left[ \frac{\partial f_1}{\partial x_1} \dot{x}_1 + \frac{\partial f_1}{\partial x_2} \dot{x}_2 \right] + \frac{\partial V}{\partial f_2} \left[ \frac{\partial f_2}{\partial x_1} \dot{x}_1 + \frac{\partial f_2}{\partial x_2} \dot{x}_2 \right] \right\}$$

where

$$\frac{\partial V}{\partial f_1} = 2a_{11} f_1 + 2a_{12} f_2,$$

$$\frac{\partial V}{\partial f_2} = 2a_{12} f_1 + 2a_{22} f_2,$$

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_2}{\partial x_1} = -Kf', \quad \frac{\partial f_2}{\partial x_2} = -w .$$

Thus, the conditions for asymptotic stability are as follows:

if $a_{11} = a_{12} > 0$, $a_{22} = \gamma a_{12}$, then

$\gamma > 1$, $w\gamma > 1$ , and

$$\frac{\gamma(w+1) - 2 - 2\sqrt{(\gamma - 1)(\gamma w - 1)}}{\gamma^2} < Kf' < \frac{\gamma(w+1) - 2 + 2\sqrt{(\gamma - 1)(\gamma w - 1)}}{\gamma^2}$$

## Example 3, [1]    Second Order, Nonlinear System

The system is described by

$$\dot{x}_1 = f_1(x_1) + f_2(x_2),$$

$$\dot{x}_2 = x_1 + ax_2,$$

$$f_1(0) = f_2(0) = 0,$$

$f_1$, $f_2$ are differentiable.

Therefore,

$$\hat{F} = \begin{array}{|c|c|} \hline 2 f_1'(x_1) & 1 + f_2'(x_2) \\ \hline 1 + f_2'(x_2) & 2a \\ \hline \end{array} .$$

To satisfy the hypotheses of Krasovskii's theorem, we let

(1)

$$4 a \left[\frac{df1}{dx_1}\right] - \left[1 + \left(\frac{df2}{dx_2}\right)^2\right]^2 \geq \epsilon^2 > 0 \qquad \text{for any } \underline{x},$$

(2)

$$\frac{df1}{dx_1} \leq -\epsilon < 0 \quad \text{for all} \quad x_1.$$

Example 4, [3]                    Second Order, Nonlinear

We have a second order system defined by

$$\dot{x}_1 = f_1(x_1) + g_1(x_2),$$

$$\dot{x}_2 = f_2(x_1) + g_2(x_2).$$

Thus,

$$\hat{\underline{F}} = \begin{array}{|c|c|} \hline 2f_1' & f_2' + g_1' \\ \hline f_2' + g_1' & 2g_2' \\ \hline \end{array}$$

The characteristic equation for $\hat{\underline{F}}$ is

$$\lambda^2 - 2(f_1' + g_2')\lambda + 4f_1' g_2' - \left\{g_1' + f_2'\right\}^2 = 0.$$

The real parts of the eigenvalues are negative if

(1) $g_1' = f_2'$ for $\underline{x} \neq \underline{0}$,

(2) $f_1' + g_2' < 0$ for $\underline{x} \neq 0$.

Therefore, the system will be completely stable if (1) and (2) hold.

The next three examples are systems which can not be analyzed by Krasovskii's theorem but are applications of the more general theorems of Mangasarian. These examples are nonautonomous and have right-hand sides which are not everywhere differentiable.

<u>Example 5,</u> [6]          <u>Second Order, Nonautonomous</u>

In state variable notation, the defining equations are

$$\dot{x}_1 = x_2 ,$$

$$\dot{x}_2 = - x_1 - b(t) x_2 - c(t) x_2^3 .$$

The physical application of this equation may be interpreted as the movement of a unit point mass under a unit spring force $x_1$ and under a nonlinear damping force $b (t)\dot{x}_1 + c(t) \dot{x}_1^3$ . The scalar function $\underline{x}_T \underline{f}(t, \underline{x})$ which is the test function in Mangasarian's Theorem is

$$\underline{x}_T \underline{f} = - \left\{ b(t) x_2^2 + c(t) x_2^4 \right\}.$$

This function is concave, but not strictly concave if $b(t) \geqslant 0$ and $c(t) \geqslant 0$ . Hence, by Mangasarian's Theorem 1, the equilibrium point $(x_1, x_2) = (0, 0)$ is <u>stable</u>.


<u>Example 6,</u> [6]          <u>Second Order, Autonomous</u>

The system is defined by

$$\dot{x}_1 = \begin{cases} - x_1 - x_1 x_2 , & \text{for } x_2 \leq 1 , \\ - x_1 - x_1 x_2^3 , & \text{for } x_2 > 1 , \end{cases}$$

$$\dot{x}_2 = \begin{cases} - x_2 + x_1^2 , & \text{for } x_2 \leq 1 , \\ - x_2 + x_1^2 x_2^2 , & \text{for } x_2 > 1. \end{cases}$$

For this system we have $\underline{f}(\underline{0}) = \underline{0}$. Also, the test function, $\underline{x}_T \underline{f}(\underline{x})$ is

$$\underline{x}_T \underline{f} (\underline{x}) = - x_1^2 - x_2^2 ,$$

which is strictly concave. Hence by Theorem 2 of Mangasarian, the equilibrium point (0, 0) is uniformly asymptotically stable in the large. Krasovskii's

result can not be applied here because $\underline{f}$ is nondifferentiable along the line $(x_1, 1)$ in phase space.

### Example 7, [6]    Nonautonomous System

The system is described by

$$\dot{x}_1 = \begin{cases} (1 + \gamma \sin t)(- x_1 + x_1 x_2^2 ), & x_1^2 + x_2^2 \leq 1 , \\ (1 + \gamma \sin t)(- x_1 - x_1 x_2^2 ), & x_1^2 + x_2^2 > 1 , \end{cases}$$

$$\dot{x}_2 = \begin{cases} (1 + \gamma \sin t)(- x_2 - x_1^2 x_2) , & x_1^2 + x_2^2 \leq 1 , \\ (1 + \gamma \sin t)(- x_2 + x_1^2 x_2) , & x_1^2 + x_2^2 > 1 , \end{cases}$$

where $0 < \gamma < 0.9$ . For this example $\underline{f}(t, \underline{0}) = \underline{0}$. Also, $\underline{f}(t, \underline{x})$ is discontinuous along the circle $x_1^2 + x_2^2 = 1$. However, the scalar function

$$\underline{x}_T \underline{f}(t, \underline{x}) = - (1 + \gamma \sin t)(x_1^2 + x_2^2 )$$

is strictly concave in $\underline{x}$ for $0 \leq t < \infty$ , and the

$$\limsup_{t \to \infty} \left\{ -(1 + \gamma \sin t)(x_1^2 + x_2^2 ) \right\} \leq - \frac{1}{10} (x_1^2 + x_2^2 ) < 0 ,$$

for $\underline{x} \neq \underline{0}$ . Hence, by Theorem 2 of Mangasarian, the point $(0,0)$ is a uniformly, asymptotically stable point in the large.

Example 8, [9]          Steady-State Condition - With Inputs

The system is defined by

$$\dot{x}_1 = x_2 - ax_1 (x_1^2 + x_2^2) + i_1(t) \quad ,$$

$$\dot{x}_2 = -x_1 - ax_2 (x_1^2 + x_2^2) + i_2(t) \quad .$$

Assume that the inputs settle eventually down to constant inputs $i_1$ and $i_2$.

The problem is to determine if the system is uniformly asymptotically stable

in the large about its equilibrium point wherever that may be.  The matrix

$$\underline{A} = \left[ -\frac{\partial f_i}{\partial x_j} \right] \quad \text{is}$$

$$\underline{A} = \begin{array}{|c|c|} \hline 3ax_1^2 & 2ax_1x_2 - 1 \\ \hline 2ax_1x_2 + 1 & 3ax_2^2 \\ \hline \end{array} \quad .$$

Choose the kinetic Liapunov function to be of the form $K = \underline{\dot{x}}_T \underline{\dot{x}} = \dot{x}_1^2 + \dot{x}_2^2$ .

Thus,

$$\dot{K} = -\underline{\dot{x}}_T \left\{ \underline{A}_T + \underline{A} \right\} \underline{\dot{x}} \quad ,$$

where

$$\underline{A}_T + \underline{A} = \begin{array}{|c|c|} \hline 6\,ax_1^2 & 4ax_1x_2 \\ \hline 4\,ax_1x_2 & 6ax_2^2 \\ \hline \end{array} \quad .$$

The matrix $\underline{A}_T + \underline{A}$ is positive definite except at $(0,0)$ if $a > 0$; thus, the

system is uniformly asymptotically stable in the large about its equilibrium

point.

Example 9, [9]                    Systems with a Nonlinear Gain Element

Systems with nonlinear gain element can be described in state variable $\underline{x}$

notation by the equation

$$\dot{\underline{x}} = - \underline{A}_o \underline{x} - \underline{b} \; f(x_1) + \left[ \underline{g}_o \; r(t) + g_1 \dot{r}(t) + \ldots \right] \; .$$

For inputs in which the square bracketed term is constant, a steady state

equilibrium point $\underline{x}_e$ exists. We will consider the stability of this $\underline{x}_e$, which

is the equilibrium point of

$$\dot{x} = - \underline{A}_o \underline{x} - \underline{b} \; f(x_1) + \text{constant}.$$

Differentiating this equation gives

$$\ddot{x} = - \underline{A}_o \dot{\underline{x}} - f'(x_1) \underline{A}_1 \dot{\underline{x}} \; ,$$

where $\underline{A}_1$ is a matrix with $\underline{b}$ as its first column and zeros everywhere else. Let

U and L be the upper and lower bounds of $f'(x_1)$:

$$L \le f'(x_1) \le U.$$

From Chang's theoretical discussion, the general system will be uniformly

asymptotically stable in the large if a common orientation can be found for

$(\underline{A}_o + L \underline{A}_1)$ and $(\underline{A}_o + U\underline{A}_1)$.


First Order System

The defining equation is

$$\dot{x} = - ax - f(x) + r(t)$$

where $r(t) \longrightarrow$ constant, "sufficiently fast." The matrices $\underline{A}_0$ and $\underline{A}_1$

are first order and equal(a) and (1), respectively. The sufficient condition

for uniformly asymptotic stability in the large is that $(a + f'(x)) > 0$.

This is also the necessary condition if the system is to be stable for every

$x_e$.

## Second Order System

Consider the system

$$\dot{x}_1 = x_2 ,$$

$$\dot{x}_2 = -ax_2 - f(x_1),$$

where $L \leq f'(x_1) \leq U$. A common orientation exists for the matrices

| 0 | -1 |
|---|----|
| L | a  |

and

| 0 | -1 |
|---|----|
| U | a  |

if

   i)  $L > 0$, $a > 0$

and

   ii)  $U < (a + \sqrt{L})^2$ .

Condition (i) is necessary for local stability, while conditions (ii) and (i) are both sufficient for uniform asymptotic stability in the large.

In this case Chang can easily show that his kinetic Liapunov function is too conservative. Assume $f(0) = 0$. Consider

$$V = \int_o^{x_1} f(x) \, dx + \frac{1}{2} (ax_1 + x_2)^2$$

as a candidate for a Liapunov function. The time derivative is

$$\dot{V} = f(x_1) \dot{x}_1 + (ax_1 + x_2)(a\dot{x}_1 + \dot{x}_2)$$

$$= x_2 f(x_1) + (ax_1 + x_2)(ax_2 - ax_2 - f(x_1))$$

$$= - ax_1 f(x_1).$$

Thus, V is a Liapunov function is $a > 0$ and $x_1 f(x_1) > 0$, $x_1 \neq 0$ . These conditions imply the conditions in (i), but, (ii) is not necessary.

<u>Example 10, 10</u>　　　　　　　　　<u>Third Order System</u>

The third order system is given by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -(x_1 + cx_2)^3 - bx_3,$$

where

$$\underline{B}(\underline{x}) =$$

| 0 | 1 | 0 |
|---|---|---|
| 0 | 0 | 1 |
| $-3(x_1 + cx_2)^2$ | $-3c(x_1 + cx_2)^2$ | $-b$ |

.

This matrix $\underline{B}(\underline{x})$ has the same form as the $\underline{B}$ in the linear case, but with

variable elements.  The third-order matrix corresponding to $\underline{A}_2$ in the table

under "Ingwerson's Work" is

$$\underline{A}_2(\underline{x}) =$$

| $3b(x_1 + cx_2)^2$ | $3(x_1 + cx_2)^2$ | 0 |
|---|---|---|
| $3(x_1 + cx_2)^2$ | $b^2 + 3c(x_1 + cx_2)^2$ | b |
| 0 | b | 1 |

.

This is the same as $\underline{A}_2$ for the linear case except certain constant elements

are replaced by variables.

Next, matrix $\underline{A}_2(x_i, x_j)$ is obtained from $\underline{A}_2(\underline{x})$ by letting certain variables

vanish:

$$\underline{A}_2(x_i, x_j) =$$

| $3bx_1^2$ | $3(x_1 + cx_2)^2$ | 0 |
|---|---|---|
| $3(x_1 + cx_2)^2$ | $b^2 + 3c^3 \cdot x_2^2$ | b |
| 0 | b | 1 |

.

Performing the integration

$$\underline{V}\underline{V} = \int_{\underline{o}}^{\underline{x}} \underline{A}_2(x_i, x_j)\, d\underline{x} ,$$

gives

$$\nabla V = \begin{vmatrix} bx_1^3 + 1/c\,(x_1 + cx_2)^3 - x_1^3/c \\ (x_1 + cx_2)^3 + b^2 x_2 + bx_3 \\ bx_2 + x_3 \end{vmatrix}$$

The Liapunov function V is found by evaluating the line integral of $\nabla V$ along any path of integration:

$$V = \int_{\underline{0}}^{\underline{x}} \nabla V \cdot d\,\underline{x} = \frac{bx_1^4}{4} + \frac{(x_1 + cx_2)^4}{4c} - \frac{x_1^4}{4c} +$$

$$+ \frac{b^2 x_2^2}{2} + bx_2x_3 + x_3^2/2 .$$

When $b > 0$, $c > 0$, $bc-1 > 0$, then $V > 0$ and $\nabla V = \underline{0}$ only when $\underline{x} = \underline{0}$. The time derivative of V is given by

$$\dot{V} = \nabla V_T \underline{x} = -(bc-1)(3x_1^2 + 3cx_1x_2 + c^2 x_2^2) x_2^2 .$$

$\dot{V}$ is negative semidefinite when $b > 0$, $c > 0$, and $bc-1 > 0$. In this particular case, the conditions are also necessary for the asymptotic stability of the system.

Example 11, [10]                    Nonlinear Compensation

The physical system considered here is an undamped second-order system which is made asymptotically stable by regulating the gain of an amplifier. The equations of motion in state-variable form are

$$\dot{x}_1 = x_2 - x_{2o}$$

$$\dot{x}_2 = \frac{b_o}{J}(x_1 - x_{10}) - \frac{b_1}{J}(x_1 - x_{10})^2 (x_2 - x_{20}).$$

The terms $x_{10}$ and $x_{20}$ are time varying inputs which are held constant after some time $t_1$. Then for $t > t_1$, the substitution, $y_1 = x_1 - x_{10}$ and $y_2 = x_2 - x_{20}$, reduces the above equation to

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = -\frac{bo}{J} y_1 - \frac{b_1}{J} y_1^2 y_2.$$

If these equations are asymptotically stable in the large, then the original system is stable for all inputs.

The matrix $\underline{B}(\underline{y})$ is

$$\underline{B} = \begin{array}{|c|c|}
\hline
0 & 1 \\
\hline
-\dfrac{bo}{J} - \dfrac{2b_1}{J} y_1 y_2 & -\dfrac{b_1}{J} y_1^2 \\
\hline
\end{array}.$$

The first matrix in Ingwerson's table is used, that is

$$\underline{A} = \begin{array}{|c|c|}
\hline
\dfrac{b_o}{J} + \dfrac{2b_1}{J} y_1 y_2 & 0 \\
\hline
0 & 1 \\
\hline
\end{array}.$$

The matrix $\underline{A}(y_i, y_j)$ becomes

$$\underline{A}(y_i, y_j) = \begin{array}{|c|c|}
\hline
bo/J & 0 \\
\hline
0 & 1 \\
\hline
\end{array} \quad ;$$

thus, $\underline{\nabla V}$ becomes

$$\underline{\nabla V} = \int_{\underline{0}}^{\underline{x}} \underline{A}(y_i, y_j) \, d\underline{x} = \begin{array}{|c|}
\hline
\dfrac{bo\ y_1}{J} \\
\hline
y_2 \\
\hline
\end{array}.$$

Therefore, $V$ and $\dot{V}$ become

$$V = \left\{ \frac{bo}{2J} y_1^2 + \frac{1}{2} y_2^2 \right\},$$

$$\dot{V} = - \frac{b_1}{J} \, y_1^2 \, y_2^2 \, .$$

If $b_0 > 0$, $b_1 > 0$ and $J > 0$, then $V$ is positive definite and $\dot{V}$ is negative semidefinite and the only trajectory of the system for which $\dot{V}$ is identically zero is $\underline{y} = \underline{0}$. Thus, the system is asymptotically stable in the large.

<u>Example 12, [10]</u>    <u>Discontinuous System</u>

The physical system is a linear second order switching system whose actuating signal is switched positive or negative according to the sign of the linear switching criterion $x_1 + cx_2$. The equations of motion are

$$\dot{x}_1 = x_2 \, ,$$

$$\dot{x}_2 = - K \omega^2 \, \text{sgn} \, (x_1 + cx_2) - \omega^2 x_1 - 2D \omega x_2 \, .$$

From the theory of generalized functions, we know that the derivative of sgn x is twice the Dirac delta function, $\delta(x)$. Thus, the Jacobian matrix of the above system is

$$\underline{B}(\underline{x}) = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -\omega^2 - 2K \omega^2 \, \delta \, (x_1 + cx_2) & -2D \omega - 2Kc \omega^2 \, \delta \, (x_1 + cx_2) \\ \hline \end{array}$$

In this case Ingwerson considers a linear combination of the $\underline{A}$-matrices from "his table", namely, $\mu \underline{A}_1 + \underline{A}_2$. The first row, first column element of $\mu \underline{A}_1 + \underline{A}_2$ must be modified so that $\delta^2(x)$ does not occur. Thus, from the modified $\mu \underline{A}_1 + \underline{A}_2$ we get the matrix $\underline{A}(x_i, x_j)$ in the following form:

$$\underline{A}(x_i, x_j) = \begin{array}{|c|c|} \hline \left(4D^2 + 1 + \mu \right) \omega^2 + 2K \omega^2 \, (2cD \omega + 1 + \mu) \, \delta(x_1) & 2D \omega + 2Kc \omega^2 \delta(x_1 + cx_2) \\ \hline 2D \omega + 2Kc \omega^2 \, \delta \, (x_1 + cx_2) & 1 + \mu \\ \hline \end{array}$$

Integration of this matrix gives $\nabla V$ in the form:

$$\nabla V = \begin{array}{|l|} \hline (4D^2 + 1 + \mathcal{H})\omega^2 x_1 + K\omega^2(2cD\omega + \mathcal{H}) \text{ sgn } x_1 + 2D\omega x_2 + K\omega^2 \text{sgn}(x_1 + cx_2) \\ \hline 2D\omega x_1 + Kc\omega^2 \text{sgn}(x_1 + cx_2) - Kc\omega^2 \text{sgn } cx_2 + (1 + \mathcal{H})x_2 \\ \hline \end{array}$$

The time derivative of V is equal to $\nabla V_T \dot{x}$, which is a very complicated expression. To simplify $\dot{V}$, Ingwerson first let $\mathcal{H} = \omega^2 c$ which makes certain terms in $\dot{V}$ nonpositive. Also, it can be observed that the (sgn $x_1$) and (sgn $Cx_2$) terms in $\dot{V}$ complicate matters. These terms are dropped without interfering with the integrability of $\nabla V$. Finally, we have

$$V = (4D^2 + 1 + \omega^2 c^2)\left[\frac{x_1^2}{c}\right] + D\omega x_1 x_2 + K\omega^2(x_1 + cx_2)$$

$$\cdot \text{sgn}(x_1 + cx_2) + (1 + \omega^2 c^2) x_{2/2}^2,$$

$$\nabla V = \begin{array}{|l|} \hline (4D^2 + 1 + \omega^2 c)\omega^2 x_1 + 2D\omega x_2 + K\omega^2 \text{sgn}(x_1 + cx_2) \\ \hline 2D\omega x_1 + Kc\omega^2 \text{sgn}(x_1 + cx_2) + (1 + \omega^2 c^2) x_2 \\ \hline \end{array}$$

and

$$\dot{V} = \nabla V_T \dot{x} = -2D\omega^3(x_1^2 + c^2 x_2^2) - K^2 c\omega^4 +$$

$$- K\omega^3(2D + c\omega)(x_1 + cx_2) \text{ sgn}(x_1 + cx_2).$$

Therefore, V is positive definite and $\dot{V}$ is negative, giving us asymptotic stability in the large.

Example 13, [10]     Third Order - Nonlinear Case

The third order equation

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + f(y)y = 0$$

is asymptotically stable in the large if the roots of its characteristic

equation, calculated as if $f(y)$ were constant, have negative real parts. The proof of this statement is given below by the application of Ingwerson's method to this problem.

In order to prove that the above conclusions are valid we choose the matrix $\underline{C}$ in the equation

$$\underline{B}_T \underline{A} + \underline{A} \ \underline{B} = - \ \underline{C}$$

to be of the form

$$
\underline{C} \ = \
\begin{array}{|c|c|c|}
\hline
2a_2(yf' + f) & 0 & 2(yf' + f) \\
\hline
0 & 0 & 0 \\
\hline
2(yf' + f) & 0 & 2a_1 \\
\hline
\end{array}
$$

From the usual form of the equivalent system of first order equations, we have the matrix $\underline{B}$ given as

$$
\underline{B} \ = \
\begin{array}{|c|c|c|}
\hline
0 & 1 & 0 \\
\hline
0 & 0 & 1 \\
\hline
-f(y) & -a_2 & -a_1 \\
\hline
\end{array} \ .
$$

Thus, the $\underline{A}$ can be determined and is then reduced to the following matrix $\underline{A}^*$.

$$
\underline{A}^* \ = \
\begin{array}{|c|c|c|}
\hline
a_2^2 + a_1(yf' + f) & a_1 a_2 & a_2 \\
\hline
a_1 a_2 & a_1^2 + a_2 & a_1 \\
\hline
a_2 & a_1 & 2 \\
\hline
\end{array} \ .
$$

Integrate $\underline{A}^*$ to get $\underline{\underline{V}V}$, integrate $\underline{\underline{V}V}$ to get $V$, and form $\underline{\underline{V}V}_T \underline{\dot{x}}$ to get $\dot{V}$; thus, we have:

$$
\underline{\underline{V}V} \ = \
\begin{array}{|c|}
\hline
a_2^2 y + a_1 f(y) \ y + a_1 a_2 \dot{y} + a_2 \ddot{y} \\
\hline
a_1 a_2 y + (a_1^2 + a_2) \ \dot{y} + a_1 \ddot{y} \\
\hline
a_2 y + a_1 \dot{y} + 2 \ddot{y} \\
\hline
\end{array} \ ,
$$

$$V = \left[a_2^2/_2\right] y^2 + a_1 \int_0^y f(y)y \, d \, y + a_1 a_2 y \, \dot{y} + a_2 y \, \ddot{y} +$$

$$+ (a_1^2 + a_2) \, \dot{y}^2/_2 + a_1 \dot{y} \ddot{y} + \ddot{y}^2 \; ;$$

$$\dot{V} = -\left\{a_2 f(y) \, y^2 + 2f(y) \, y \, \ddot{y} + a_1 \ddot{y}^2\right\}.$$

Therefore, V is positive definite if $a_1 > 0$, $a_2 > 0$, $f(y) > 0$; and $\dot{V}$ is nonpositive if $a_1 > 0$, $a_2 > 0$, $f(y) > 0$ and $a_1 a_2 - f(y) > 0$. These are the Routh-Hurwitz inequalities for the characteristic equation defined for this system where $f(y)$ replaces the constant $a_3$. From linear theory these conditions are necessary and sufficient for the characteristic roots to have negative - real parts. For this nonlinear case we have thus obtained sufficient stability conditions, but not necessary conditions.

Example 14, [13, 14]          Third Order Example

This problem deals with a third order linear plant with a nonlinear element. The gain of the nonlinearity is the sum of a linear and a cubic term. For large error signals, the nonlinear gain can cause instability if the velocity feedback coefficient, $C_2$, is not sufficiently large. The system is defined by the following set of equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = - b_1 x_3 - (b_2 + c_2 b_3) x_2 - b_3 \, x_1 - b_4 (x_1 + c_2 x_2)^2.$$

For this system, we will only summarize the results of Ingwerson and Rodden and not consider the detailed analysis of the problem.

(1) Ingwerson has shown that the system is globally asymptotically stable if $b_1 > 0$, $b_2 > 0$, $b_3 > 0$, $b_4 > 0$, $C_2 > 0$ and $b_1 C_2 > 1$.

(2) For no damping ($C_2 = 0$), Ingwerson analytically derived and investigated the fourth degree Liapunov function:

$$V = b_1b_3 \, x_1^2 + \frac{b_1b_4}{2} \, x_1^4 + 2 \, b_3x_1x_2 + 2b_4x_1^3 \, x_2 +$$

$$+ \; (b_1^2 + b_2) \, x_2^2 + 2b_1x_2x_3 + x_3^2 \; ,$$

whose derivative is

$$\dot{V} = - \, 2 \left\{ b_1b_2 - b_3 - 3 \, b_4 \, x_1^2 \right\} \, x_2^2 \, .$$

For $b_1b_2 - b_3 > 0$, $\dot{V} > 0$ outside the region bounded by the planes

$$x_1 = \pm \left\{ \frac{b_1b_2 - b_3}{3b_4} \right\}^{\frac{1}{2}} .$$

The stability domain is the minimum $V$-surface which is tangent to these planes. The tangent points are given by the following:

$$x_2 = - \left[ \frac{x_1}{b_2} \right] (b_3 + b_4x_1^2) \; , \quad x_3 = \left[ \frac{b_1x_1}{b_2} \right] (b_3 + b_4 \, x_1^2) \; ,$$

and

$$x_1 = \pm \left\{ \frac{b_1b_2 - b_3}{3b_4} \right\}^{\frac{1}{2}}$$

(3) Rodden, for $b_1 = b_2 = 2$, $b_3 = b_4 = 1$, estimated the region of asymptotic stability for the above system with the use of an electronic computer and the formulation of Ingwerson.

### Example 15, [16]     Second Order Systems

Consider the system defined by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \epsilon (1 - x_1^2 + x_1^4) \, x_2 - x_1^3 \, .$$

The simplest form of a definite function is

$$V = \underline{x}_T \underline{A} \ (\underline{x}) \ \underline{x} = \underline{x}_T \begin{bmatrix} a_1(x_1) & 0 \\ 0 & 1 \end{bmatrix} \underline{x} \ .$$

Let us consider the form, $\psi \ (\underline{x})$, defined by

$$\psi(\underline{x}) = \underline{x}_T \left\{ \underline{B}_T \underline{A} + \underline{A} \ \underline{B} \right\} \underline{x} \ ,$$

where $\underline{B}$ is defined by $\underline{\dot{x}} = \underline{B} \ \underline{x}$. For this system, $\psi \ (\underline{x})$ takes the form

$\psi(\underline{x}) = 2a_1(x_1) \ x_1 x_2 + 2x_2 \left\{ \in (1 - x_1^2 + x_1^4) \ x_2 - x_1^3 \right\}$ . To find

the limit surface, we set $\psi(\underline{x})$ equal to zero. Thus, for $\psi(\underline{x}) = 0$ ,

one has

$$x_2 = 0, \quad x_2 = \left\{ \frac{x_1^3 - a_1(x_1)x_1}{\in (1 - x_1^2 + x_1^4)} \right\} .$$

We now let these two curves, for which $\psi \ (\underline{x}) = 0$ , coincide.

Therefore, we have

$$x_2 = 0 = \left\{ \frac{x_1^3 - a_1(x_1)x_1}{\in (1 - x_1^2 + x_1^4)} \right\} ,$$

or $a_1(x_1) = x_1^2$. When the above two curves coincide the equilibrium solution

$\underline{x} = \underline{0}$ will either be globally stable or unstable.

From the above procedure we have obtained the type of $a_1(x_1)$ function

which is required, but for more flexibility we replace the $a_1(x_1)$ in V by

a constant, $b_1$, times $x_1^2$ . Thus, our candidate for a Liapunov function

is $V = b_1 x_1^4 + x_2^2$, which is positive definite if $b_1 > 0$. The time

derivative is

$$\dot{V} = 4b_1 x_1^3 x_2 + 2 \in x_2^2 (1 - x_1^2 + x_1^4) - 2x_1^3 x_2 \ ;$$

where for $b_1 = \frac{1}{2}$, we have

$$\dot{V} = 2 \in x_2^2 (1 - x_1^2 + x_1^4) .$$

Thus, $\dot{V}$ is semidefinite and is not identically equal to zero on any non-trivial solution of the original system. In conclusion, the system is globally asymptotically stable if $\in < 0$ and globally completely unstable if $\in > 0$.

## Example 16, [16]      Van der Pol's Equation

Let us consider van der Pol's equation with $\in = 1$ :

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \in (1 - x_1^2) x_2 - x_1 .$$

By the following choice of V we can show that the system is unstable inside the circle $x_1^2 + x_2^2 = 1$ :

$$V = a_1 x_1^2 + a_2 x_2^2 .$$

The function, $\gamma$, defined in example 15, becomes

$$\gamma(\underline{x}) = 2a_1 x_1 x_2 + 2a_2 x_2^2 - 2a_2 x_1^2 - 2a_2 x_1 x_2 .$$

As before, we let $\gamma(\underline{x}) = 0$ and the result is that $x_2 = 0$ and

$$x_2 = \left\{ \frac{(a_2 - a_1) x_1}{a_2 (1 - x_1^2)} \right\} .$$   The curves coincide if $a_1 = a_2$.  Thus, let $a_1 = a_2 = 1$ in V.  This gives

$$V = x_1^2 + x_2^2$$

and

$$\dot{V} = 2 x_2^2 (1 - x_1^2) .$$

Therefore, from instability theorems, the van der Pol equation is unstable inside the circle $x_1^2 + x_2^2 = 1$.

As another generating V-function we choose

$$V = a_1(x_1) x_1^2 + 2a_{12} (x_1) x_1 x_2 + a_2 x_2^2 \ ,$$

where

$$\gamma (\underline{x}) = x_2^2 (a_{12} + a_2 - a_2 x_1^2) - a_{12} x_1^2 +$$

$$+ x_1 x_2 (a'_1 - a_2 + a_{12} + a_{12} x_1^2) \ .$$

By solving the equation $\gamma(\underline{x}) = 0$ and constraining the two surfaces, given by

solutions of this equation, to coincide and assuming that the functions $a_{ij}(x_1)$ are

polynomials in $x_1$, the following expressions for $a_1(x_1)$ and $a_{12}(x_1)$ are obtained:

$$a_1(x_1) = a_2 x_1^4 - 2a_2 x_1^2 + 2a_2,$$

$$a_{12}(x_1) = a_2 x_1^2 - a_2.$$

Before substituting these expressions for $a_{12}$ and $a_1$ into the V-function,

the following **constants** are introduced:

$$a_1(x_1) = b_1 x_1^4 - b_2 x_1^2 + 2a_2,$$

$$a_{12}(x_1) = \tfrac{1}{2} b_3 x_1^2 - a_2.$$

Thus, our V-function becomes

$$V = b_1 x_1^6 - b_2 x_1^4 + b_3 x_1^3 x_2 + 2a_2 x_1^2 +$$

$$- 2a_2 x_1 x_2 + a_2 x_2^2 \ ,$$

where the corresponding $\overset{\bullet}{V}$ with respect to the system is given by

$$\overset{\bullet}{V} = (6b_1 - b_3) x_1^5 x_2 + (b_3 - 4b_2 + 2a_2) x_1^3 x_2 +$$

$$+ (3 b_3 - 2a_2) x_1^2 x_2^2 - b_3 x_1^4 + 2a_2 x_1^2.$$

Rewriting $\overset{\bullet}{V}$ gives

$$\overset{\bullet}{V} = \Theta(\underline{x}) \ g \left\{ \xi (\underline{x}) \right\} = x_1^2 \left\{ (6b_1 - b_3) x_1^3 x_2 + \right.$$

$$\left. + (b_3 - 4b_2 + 2a_2) x_1 x_2 + (3 b_3 - 2a_2) x_2^2 - b_2 x_1^2 + 2a_2 \right\} \ ,$$

where the expression in the brackets, $\{\,.\,\}$, is $g\left\{\xi(\underline{x})\right\}$ . In order to simplify the form $g\left\{\xi(\underline{x})\right\}$ we set $b_3 = 6\,b_1$. Consequently, the curve $\xi(\underline{x}) = 0$ is defined by

$$3\,b_1^2\,x_1^2 - (3b_1 - 2b_2 + a_2)\,x_1 x_2 + (a_2 - 9b_1)\,x_2^2 = a_2,$$

and the V-function becomes

$$V = b_1 x_1^6 - b_2 x_1^4 + 6b_1 x_1^3 x_2 + 2a_2 x_1^2 - 2a_2 x_1 x_2 + a_2 x_2^2 .$$

Thus, if one chooses $b_1$, $b_2$ and $a_3$ in the following way the curve $\xi(\underline{x}) = 0$ is closed (definite) and the family of curves, V = constant, is a family of closed (definite) curves:

$$6b_1(2a_2 - 18b_1) - (3b_1 - 2b_2 + a_2)^2 > 0 ,$$

$$b_1 > 0,\ a_2 > 9b_1.$$

The Poincare - Bendixon theory states that there exists at least one stable limit cycle in the annular region defined by two curves of the family, V = constant; namely, the inscribed and circumscribed curves to the closed curve $\xi(\underline{x}) = 0$. By numerical techniques, this stable limit cycle can be approximated "arbitrarily close". Also, it can be proved that outside the circumscribed curve the system is asymptotically stable.

Example 17, [16]      Third Order System

This is an application of Szego's technique to prove the existence of local stability for systems with only one critical point. The system is defined by:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 3x_3 - 2x_2 + x_1^3 ,$$

which has a critical point at $\underline{x} = \underline{0}$. The form of the V-function is assumed to be

$$V = a_{11}\, x_1^2 + a_{22}\, x_2^2 + x_3^2 + 2a_1a_2\, x_1x_2 +$$

$$+ 2a_{23}\, x_2x_3 + 2a_{13}x_1x_3,$$

where $a_{ij} = a_{ij}\,(x_1, x_j)$. The corresponding $\dot{V}(\underline{x})$ for this third order system is

$$\dot{V}\,(\underline{x}) = x_3^2\,(2a_{23} - 6) + x_3 \left\{ 2a_{22}x_2 + 2a_{12}\, x_1 - 6a_{23}x_2 + \right.$$

$$\left. - 6x_2 - 2x_1^3 + 2a_{13}\, x_2 - 6a_{13}\, x_1 \right\} +$$

$$+ 2 \left\{ a_{11}\, x_1x_2 + a_{12}\, x_2^2 - 2a_{23}\, x_2^2 + \right.$$

$$\left. - 2a_{13}\, x_1x_2 - a_{13}x_1^4 - 2a_{23}x_1^3\, x_2 \right\}.$$

The conditions which make $\dot{V}$ indefinite on a closed surface are

$$a_{13} = 0, \quad a_{23} = 0, \quad a_{12} = x_1^2, \quad a_{22} = 11.$$

For simplification purposes, we let

$$a_{11} = 3/2 \; x_1^2.$$

Therefore, we get

$$V = 3/2\, x_1^4 + 11x_2^2 + x_3^2 + 2x_1^3\, x_2 + 6x_2x_3$$

where

$$\dot{V} = 6\, x_2^2\,(x_1^2 - 2).$$

The V-function is positive and closed between the planes $\left| x_1 \right| = \sqrt{3}$ and $\dot{V}$ is negative semidefinite and not equal to zero on a trajectory of the system between the planes $\left| x_1 \right| = \sqrt{2}$. $\dot{V}$ is positive semidefinite outside $\left| x_1 \right| = \sqrt{2}$. Thus, the system is stable in the

region defined by the closed surface

$$V = 3/2\ x_1^{\ 4} + 11\ x_2^{\ 2} + x_3^{\ 2} + 2x_1^{\ 3}\ x_2 + 6x_2 x_3 \qquad \bigg| \qquad |x_1| \leq \sqrt{2}$$

and unstable according to Chetaev's theorem outside the planes $\quad |x_1| = \sqrt{3}$ .

REFERENCES

(1) Kalman, R. E., and Bertram, J. E., "Control System Analysis and Design via the Second Method of Liapunov," Trans. ASME, Ser. D., Vol, 82, 1960, pp 371-393.

(2) Krasovskii, N. N., "On the Global Stability of a System of Nonlinear Differential Equations," P.M.M., Vol. 18, 1954, pp 735-737.

(3) Geiss, G., The Analysis and Design of Nonlinear Control Systems via Liapunov Direct Method, Grumann Aircraft Eng. Corp., Research Dept. Rep., G. R. D. No. 121, 1964.

(4) Cunningham, W. J., "An Introduction to Liapunov's Second Method, "Proc. of the Workship Session in Liapunov's Second Method, AIEE, Mich. Press, 1962, pp 1-38.

(5) Gibson, J. E., Nonlinear Automatic Control, McGraw Hill, New York, 1963, p 340.

(6) Mangasarian, O. L., "Stability Criteria for Non-linear Ordinary Differential Equations", SIAM Jour. on Control, Series A, Vol. 1, No. 3, pp 311-318.

(7) Massera, J. I., "Contributions to Stability Theory," Ann. of Math. Vol. 64, 1956, pp 182-205.

(8) Beckenbach, E. F., "Convex Functions," Bull. of Amer. Math. Soc., Vol. 54, No. 5, May 1948, pp 439-460.

(9) Chang, S. S. L., "Kinetic Liapunov Function for Stability Analysis of Nonlinear Control Systems", Jour. of Basic Eng., ASME, Vol. 83, March 1961, pp 91-94.

(10) Ingwerson, D. R., "A Modified Liapunov Method for Nonlinear Stability Analysis", IRE Trans. on Auto. Control, May 1961, pp 199-210.

(11) Vogt, W. G., Li, C. C. and Ingwerson, D. R., "Discussion of Ingwerson's Paper", IRE Trans. on Auto Control, January 1962, pp 85-88.

(12) Vogt, W. G. and Ingwerson, D. R., "Discussion of Ingwerson's Paper", IRE Trans. on Auto Control, April 1962, pp 90-92.

(13) Rodden, J. J., "Numerical Applications of Liapunov Stability Theory", IEEE - Session IX - paper 2, Auto. Control, 1964, pp 261-268.

(14) Ingwerson, D. R., A Modified Liapunov Method for Non-Linear Stability Problems, Ph. D. dissertation, Stanford University, Stanford, California, November, 1960.

(15)  Krasovskii, N. N., "On the Stability with Large Initial Disturbances", <u>PMM</u>, Vol. 21, pp 309-312, 1957.

(16)  Szego, G. P., "A Contribution to Liapunov's Second Method:  Nonlinear Autonomous Systems", <u>Journal of Basic Eng., ASME</u>, Vol. 84, Dec. 1962, pp 571-578.

WORK OF SZEGÖ AND ZUBOV

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# WORK OF SZEGO AND ZUBOV

## SUMMARY

This section considers Zubov's method for generating Liapunov functions, which are solutions of partial differential equations, along with the generalizations given by Szego. Also, included in this section is the work of Margolis and Rodden which deals with the numerical solution of Zubov's partial differential equation. A compendium of examples is given at the end of the section.

## INTRODUCTION

The first part of this section concerns itself with the discussion of the stability theorems of Zubov. In these theorems Zubov describes the partial differential equations which the Liapunov functions must satisfy to guarantee the various types of stability. The domains of asymptotic stability are also determined from the V-functions satisfying these partial differential equations. The types of problems considered in Zubov's theorems are as follows: (1) the asymptotic stability of the null solution, (2) the stability of systems with persistent perturbations, (3) the determination of analytic systems having specified domains of stability, (4) the stability of systems with homogeneous right-sides, (5) stability of systems with self-oscillations, (6) and the stability of nonautomous systems.

Next, we discuss the application of numerical techniques to the solutions of the Zubov equations. The work of Zubov, Margolis, Vogt and Rodden is considered. Zubov's theorems are listed and discussed for the second order case, with the implication that the theorems are applicable for higher order systems. The numerical results obtained by Margolis and Rodden are presented and conclusions and proposals for future work are given.

The final part of the discussion is concerned with Szegö's work. This work is in two parts. <u>One</u>, he shows that the solution of Zubov's equation can be reduced to the determination of a matrix, $\underline{A}(\underline{x})$. This technique is similar to the variable gradient method. <u>Two</u>, Szegö constructs a generalized Zubov partial differential equation. From this equation he is then able to derive meaningful stability results for a wide class of autonomous systems. He also extends this work to the determination of the stability properties of certain manifolds in the Euclidean space, $E_n$.

At the end of the section is a compendium of examples which exemplify the theories and methods of the above authors.

## ZUBOV'S WORK

In reference [1], Zubov considers an autonomous system defined by

$$\dot{\underline{x}} = \underline{f}(\underline{x}), \tag{1}$$

where $\underline{f}$ is specified in Euclidean n-space, En, and is continuously differentiable up to order $\nu \geqslant 0$. A solution of (1) exists for any $\underline{x}(0, \underline{x}_0) = \underline{x}_0$ in En. This solution will be continuously differentiable with respect to $\underline{x}_0$ up to order $\nu$. In system (1), we assume that the trivial solution $\underline{x} = \underline{0}$ exists; that is, $\underline{f}(\underline{o}) = \underline{0}$.

<u>Theorem 1</u> gives the necessary and sufficient conditions for a region A about $\underline{o}$ to be a <u>region of asymptotic stability</u> of $\underline{o}$. (This region A is an open, connected set about $\underline{x} = \underline{0}$.)

## Theorem 1

"In order that region A be a region of asymptotic stability of $\underline{x} = \underline{0}$ of the system (1), it is necessary and sufficient that there exist functions V($\underline{x}$) and $\phi(\underline{x})$, which have the following properties:

1)  V is specified and continuous in A,

2)  $\phi$ is specified and continuous in $E_n$,

3) for all $\underline{x}$ in $E_n$, $0 < V < 1$ if $\underline{x} \neq \underline{0}$,

4) for all $\underline{x}$ in $E_n$, $\phi > 0$ if $\underline{x} \neq \underline{0}$,

5) for any $\gamma_2 > 0$, there exist positive constants $\gamma_1$ and $\alpha_1$ such that

$V(\underline{x}) > \gamma_1$    for $\|\underline{x}\| \geqslant \gamma_2$ and $\phi(\underline{x}) > \alpha_1$    for $\|\underline{x}\| \geqslant \gamma_2$,

6) $V$ and $\phi \longrightarrow 0$    as    $\|\underline{x}\| \longrightarrow 0$,

7) if $\underline{Y}$ is a point on the boundary of $A$ and $\|\underline{Y}\| \neq 0$, then

$\lim V(\underline{x}) = 1$; and if $\|\underline{x}\| \longrightarrow +\infty$    for $\underline{x}$ in $A$, then $V(\underline{x}) \longrightarrow 1$.

$\left( \underline{x} \longrightarrow \underline{Y} \right)$

8) $\left[ \dfrac{dV(\underline{x}(t, \underline{x}_0))}{dt} \right]_{t=0} = - \phi(\underline{x})(1-V(\underline{x})) \sqrt{1 + \sum_{i=1}^{n} f_i^2(\underline{x})}$ ,

where the function $\phi$ can always be selected such that the function $V(\underline{x})$ is continuously differentiable over all its arguments in region $A$ up to and including order $\nu$."

Corollary 1 states that the equation of the boundary of the region of asymptotic stability, $A$, can be obtained, at least in principle. Corollary 2 deals with asymptotically stable systems on the whole.

## Corollary 1

"Consider the set of all points $\underline{x}$ such that $1 - V(\underline{x}) = \lambda$, $\underline{x}$ belonging to $A$, $0 < \lambda < 1$. This set, for each $\lambda$, represents a closed surface $S_\lambda$, which bounds a region $G_\lambda$, containing the point $\underline{x} = \underline{0}$. Surface $S_\lambda$ forms a section of the region $A$; that is, any integral curve of system (1) intersects $S_\lambda$ only once, from the outside of region $G_\lambda$ to the inside. The surface $S_0$, $V = 1$, is the boundary of region $A$, and $S_1$ coincides with the point $\underline{x} = \underline{0}$."

## Corollary 2

"The trivial solution of system (1) is called asymptotically stable in the whole, if the region A equals $E_n$; that is, if $\underline{x}(t, \underline{x}_0) \longrightarrow \underline{0}$ as $t \longrightarrow \infty$ for any $\underline{x}_0$."

The extension of theorem 1 to the concept of <u>asymptotic stability in the large</u> is considered in the next theorem.

## Theorem 2

"In order that the trivial solution of (1) be asymptotically stable in the whole, it is necessary and sufficient that there exist two functions $V_1(\underline{x})$ and $\phi(\underline{x})$, which have the following properties:

1) the functions $V_1$ and $\phi$ are specified in $E_n$ and continuous there, and
$$V_1(\underline{0}) = \phi(\underline{0}) = 0;$$

2) $V_1 > 0$ for $\underline{x} \neq \underline{0}$ and $V_1 \longrightarrow +\infty$ as $\| \underline{x} \| \longrightarrow \infty$ and
$\phi(\underline{x}) > 0$ when $\underline{x} \neq \underline{0}$;

3) for any $\gamma_2 > 0$ there exists a positive constant $\gamma_1$ such that
$\phi \geqslant \gamma_1$ when $\gamma_2 < \| \underline{x} \|$ ;

4) $\dot{V}_1 = -\phi \sqrt{1 + f_1^2 + \ldots + f_n^2}$ ."

It should be noted that the necessary and sufficient conditions for the asymptotic stability of a system in the whole were first given by <u>Barbashin</u> and <u>Krasovskii</u>. As applied to system (1), these conditions were formulated only for $\nu \geqslant 1$.

## Corollary 3

"If $\nu \geqslant 1$ and if $\underline{x} = \underline{0}$ of system (1) is asymptotically stable, then

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(\underline{x}) = -\phi(\underline{x}) \sqrt{1 + f_1^2(\underline{x}) + \ldots + f_n^2(\underline{x})} \tag{2}$$

has a unique continuously-differentiable solution, determined by $V(\underline{0}) = 0$,

specified when $\underline{x}$ belongs to A and satisfying the conditions of Theorem 1

for certain $\phi$'s. It is sufficient that $\phi(\underline{x})$ has the property

$$\int_0^\infty \phi(\underline{x}(t, \underline{x}_0)) \, dt < + \infty \qquad (3)$$

at sufficiently small $\|\underline{x}_0\|$. Thus, the $\phi$ depends on the character of

decrease in $\underline{x}(t, \underline{x}_0)$ as $t \longrightarrow \infty$, (as exemplified in the following

remarks)."

Remarks About the Above Theorems

(1) If it is known that $\|\underline{x}(t, \underline{x}_0)\| \le c \, e^{-pt}$, $c > 0$, $P > 0$,

   for sufficient small $\|\underline{x}_0\|$, then $\phi$ can be chosen such that

   $\phi(\underline{x}) \le \|\underline{x}\|^m$, $m > 0$.

(2) If it is known that $\|\underline{x}(t, \underline{x}_0)\| \le t^{-\alpha}$, $\alpha > 0$

   for $t > T$, then $\phi$ can be chosen such that $\phi(\underline{x}) \le \|\underline{x}\|^m$, $m\alpha > 1$.

(3) Examples of generating V-functions for various systems using the above

   partial differential equation for V are given in the back of this section.

   We now consider a more practical theorem of Zubov's; namely, we consider

a system of differential equations, as in (1), in which the functions $\underline{f}(\underline{x})$ are

known only approximately. That is, we are interested in whether the qualitative

estimate of $\underline{x}(t, \underline{x}_0)$ is stable with respect to small changes of the functions

$\underline{f}(\underline{x})$. Consider the system defined by

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + R(\underline{x}, t), \qquad (4)$$

where $\underline{f}(\underline{x})$ is continuously differentiable and $\underline{R}$ is continuous such that (4)

satisfies the conditions of existence and uniqueness of solutions.

Theorem 3

"If the system (1) has an asymptotically stable trivial solution, having a certain region of asymptotic stability A, it is possible to obtain for the functions $R_i(\underline{x}, t)$ an upper bound $R_i(\underline{x})$ such that $|R_i(\underline{x}, t)| < R_i(\underline{x})$ in some region about the equilibrium point. Therefore, the system in (4) has an asymptotically stable trivial solution contained in region A."

The proof of Theorem 3 relies on the results of Theorem 1. Other remarks about Theorem 3 are:

(1)  if the functions $R_i(\underline{x}, t)$ are such that $|R_i(\underline{x}, t)| < R_i(\underline{x})$ only in the region A, the statement of Theorem 3 still remains valid;

(2)  if the region A coincides with the entire space, then $\underline{x} = \underline{0}$ of system (4) is also asymptotically stable in the whole.

Another application of Theorem 1 is the following theorem giving the conditions for which it is possible to define the solutions of (1) for t belonging to $(-\infty, 0]$ .

Theorem 4

"In order for any integral curve $\underline{x}(t, \underline{x}_0)$, $\underline{x}_0$ belonging to the region A of system (1), to be defined for t belonging to $(-\infty, \infty)$, it is necessary and sufficient that all of the conditions of Theorem 1 be satisfied and that

$$\phi(\underline{x}) \sqrt{1 + f_1^2 + \ldots + f_n^2} < K < \infty, \text{ for all } t ."$$

Theorem 1 also makes it possible to establish the analytic form of the right halves of the system (1), having a previously specified region A. Thus, we specify that A is any region containing a sufficiently small neighborhood of $\underline{x} = \underline{0}$. The boundary of A is denoted by S. Region A is such that there exists a V-function with the following properties:

1) $V(\underline{0}) = 0$, $0 < V < 1$ for $\underline{x}$ in A;

2) the equation $1 - V = \lambda$, $0 < \lambda < 1$, defines a closed
surface $S_\lambda$ which bounds a region $G_\lambda$ defined by $V < 1 - \lambda$ ;

3) $V \longrightarrow + 1$ as $\| \underline{x} - \underline{x}^* \| \longrightarrow 0$, where $\underline{x}^*$ is on the boundary
of A and is not $\underline{0}$.

We determine the right halves, $\underline{f}(\underline{x})$, of (1) from the linear equations

$$\underline{P}(\underline{x})\underline{f}(\underline{x}) = \underline{q}(\underline{x}) \tag{5}$$

where $P_{1i} = \dfrac{\partial V}{\partial x_i}$ and $q_1 = - \phi_1(1-V)$ . The

function $\phi_1$ is specified in $E_n$, $\phi_1(0) = 0$, and $\phi_1 > \alpha > 0$ for
$\| \underline{x} \| > \beta > 0$ . We further assume that the functions $p_{si}$, $q_s$,
$S \geqslant 2, 3, \ldots, n$; $i=1, \ldots, n$ are so chosen that the system (5) has a solution
$\underline{f} = \underline{f}^*$ such that

$$\underline{\dot{x}} = \underline{f}^* (\underline{x}) \tag{6}$$

has a trivial solution and satisfies the conditions of existence and uniqueness
in A.

## Theorem 5

"The trivial solution of system (6) has a region of asymptotic stability A,
as defined above; and, conversely, if the system (1) has an asymptotically
stable trivial solution with a region A, then its right halves can be found
from the linear system in (5). (An example of this theorem is given in the
back of this section)."

We now consider systems of differential equations, equation (1), with
homogeneous right sides, f(x). The definition of a function which is homogeneous
of rational order $\mu = \dfrac{p}{q}$ , where q is odd, is as follows: "f($\underline{x}$) is
homogeneous of order $\mu$ provided the equality $f(c\underline{x}) = c^\mu f(\underline{x})$ holds. The

function is called <u>positive-homogeneous</u> if c $>$ 0 and $\mathcal{M}$ is arbitrary." It is known that if a homogeneous or positive-homogeneous function f is continuously differentiable of order $\mathcal{V} \geqslant 1$ over all its arguments, then it satisfies the linear partial differential equation

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i = \mathcal{M} f, \tag{7}$$

where $\mathcal{M}$ is the index of homogeneity of the function f. Also, if $\mathcal{V} \geqslant 2$, then the function $\frac{\partial f}{\partial x_i}$ is also homogeneous of order $\mathcal{M} - 1$, which can be verified by differentiating f $(c\underline{x}) = c^{\mathcal{M}} f(\underline{x})$ with respect to $x_i$.

Let us consider a system of differential equations

$$\dot{\underline{x}} = \underline{f}^{(\mathcal{M})} (\underline{x}) , \tag{8}$$

where $\underline{f}^{(\mathcal{M})}$ is continuously differentiable in all its arguments to order $\mathcal{V} \geqslant 0$ and is homogeneous of order $\mathcal{M}$, $\mathcal{M} > 0$. We assume that system (8) has a trivial solution $\underline{x} = 0$. It can be shown that if $\underline{x} = \underline{0}$ of (8) is asymptotically stable, then for a sufficiently broad class of systems the following inequality holds

$$\left\| \underline{x}(t,\underline{x}_o) \right\| \leq A_1 t^{-\alpha} \text{ for } t \geqslant T \text{ and } \left\| \underline{x}_o \right\| = 1, \tag{9}$$

where $A_1$ is a sufficiently large positive constant and $\alpha$ a sufficiently small positive quantity. The expression $\underline{x}(t, \underline{x}_o)$ is the solution of the initial value problem of the system (8).

## Theorem 6

"If the system (8) is such that its solutions satisfy the inequality (9), then there exists two functions V and W, specified in $E_n$, and having the following properties:

1) V and -W are positive definite;

2) +W is positive-homogeneous of order m, and V is positive-homogeneous of order m + 1 - $\mathcal{M}$, where m is a sufficiently large positive number;

3) $V$ is continuously differentiable along the integral curves of (8); that is, $V(\underline{x}(t, \underline{x}_0))$ has a continuous derivative $\dot{V} = W$. (From Liapunov's work we see that conditions 1, 2 and 3 are both necessary and sufficient for the asymptotic stability of the asymptotic stability of the trivial solution of system (8) satisfying inequality (9).)"

It can also be shown that if $\nu \geqslant 1$ for system (8), then $V$ and $W$ satisfy the system of partial differential equations

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i^{(\mu)} = W, \quad \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} x_i = (m - \mu + 1) V, \tag{10}$$

where $m$ is the index of homogeneity of $W$ and is a sufficiently large positive number. When $n = 2$, the function $V(x_1, x_2)$ for $\nu \geqslant 1$ can always be found in closed form. In the next paragraph we verify this statement.

## Second Order Case.

The system of equations for $V(x_1, x_2)$ is given by:

$$\frac{\partial V}{\partial x_1} f_1^{(\mu)} + \frac{\partial V}{\partial x_2} f_2^{(\mu)} = W,$$

$$\frac{\partial V}{\partial x_1} x_1 + \frac{\partial V}{\partial x_2} x_2 = (m - \mu + 1) V.$$

From these we can solve for $\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}$; that is:

$$\frac{\partial V}{\partial x_1} = AV + B, \tag{11}$$

$$\frac{\partial V}{\partial x_2} = CV + D$$

where

$$A = \frac{-(m + 1 - \mathcal{H}) f_2^{(\mathcal{H})}}{x_2 f_1^{(\mathcal{H})} - x_1 f_2^{(\mathcal{H})}} \quad ; \quad B = \frac{W x_2}{x_2 f_1^{(\mathcal{H})} - x_1 f_2^{(\mathcal{H})}}$$

$$C = \frac{-(m + 1 - \mathcal{H}) f_1^{(\mathcal{H})}}{x_1 f_2^{(\mathcal{H})} - x_2 f_1^{(\mathcal{H})}} \quad ; \quad D = \frac{W x_1}{x_1 f_2^{(\mathcal{H})} - x_2 f_1^{(\mathcal{H})}}$$

The general solution of the first equation is given by

$$V = \exp\left[\int_{x_{10}}^{x_1} A(\xi, x_2)\, d\xi\right]\left[\phi(x_2) + \int_{x_{10}}^{x_1} B(\xi, x_2) \exp\left\{-\int_{x_{10}}^{x_1} A(\eta, x_2)\, d\eta\right\} d\xi\right]$$

Choose $\phi(x_2)$ such that the function V constructed above satisfies the second equation, $\frac{\partial V}{\partial x_2} = CV + D$, and in addition, we demand that $V(0, 0) = 0$.

For simplicity, we use the following notation:

$$M_1 = \exp\left\{\int_{x_{10}}^{x_1} A(\xi, x_2)\, d\xi\right\},$$

$$M_2 = M_1 \int_{x_{10}}^{x_1} B(\xi, x_2) \exp\left\{-\int_{x_{10}}^{x_1} A(\eta, x_2)\, d\eta\right\} d\xi,$$

which gives the following formula for V,

$$V = M_1 \phi(x_2) + M_2.$$

Inserting this formula into $\frac{\partial V}{\partial x_2} = CV + D$, we obtain

$$\frac{d\phi}{dx_2} = N_1 \phi + N_2,$$

where

$$N_1 = \left[ CM_1 - \frac{\partial M_1}{\partial x_2} \right] \Big/ M_1$$

$$N_2 = \left[ CM_2 + D - \frac{\partial M_2}{\partial x_2} \right] \Big/ M_2.$$

It can be established that $\dfrac{\partial N_1}{\partial x_1} = \dfrac{\partial N_2}{\partial x_1} = 0$ .

The general solution of the above equation for $\phi$ is of the form

$$\phi(x_2) = \gamma P_1(x_2) + P_2(x_2)$$

where $\gamma$ is an arbitrary constant. If $\gamma$ is defined by

$$\gamma = - \left\{ P_2^{(o)} + M_2/M_1 \right\} \Big/ P_1(o) \qquad , \text{ for } x_1 = x_2 = 0, \text{ then the}$$

function

$$V(x_1, x_2) = \gamma M_1 P_1 + M_1 P_2 + M_2, \tag{12}$$

will satisfy system (11) and the condition that $V = 0$ when $x_1 = x_2 = 0$.

Equation (12) gives the necessary and sufficient condition for the
asymptotic stability of the trivial solution of system (8) for n = 2, and it is
asymptotically stable if and only if the function $V(x_1, x_2)$ is positive
definite. It is understood that $W(x_1, x_3)$ is negative definite and homogeneous
of order $M > \mathcal{H} - 1$ .

From reference [2] the <u>regions of attraction of self-oscillations</u> of
nonlinear systems are investigated using Zubov's partial differential
equation to determine the V-function.

Let us consider the equation

$$\underline{\dot{x}} = \underline{f}(\underline{x}) , \tag{13}$$

whose right-hand members are defined in $E_n$. We assume that the system (13)
has a periodic solution $\underline{x} = \underline{\tilde{x}}(t)$ of period T.

## Definition

"A periodic solution $\tilde{x}(t)$ of the system (13) is called <u>asymptotically</u> <u>stable</u> if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $\rho(x_0) < \delta$ , then $\rho\left[\underline{x}(t)\right] < \epsilon$ for $t > 0$ and, besides, $\rho\left[\underline{x}(t)\right] \longrightarrow 0$ as $t \longrightarrow \infty$ ."

The function $\rho$ is the <u>distance</u> from the transient process in the system (13) to the periodic behavior:

$$\rho(\underline{x}) = \inf_{t \in [o, \underline{T}]} \sqrt{\sum_{i=1}^{n} \left[x_1 - \tilde{x}_1(t)\right]^2} . \tag{14}$$

The Russian Academician A.A. Andronov called asymptotically stable periodic behaviors <u>self-oscillatory behaviors</u>. It is known that the nonlinear system (13) may have self-oscillations, while the linear systems never have them.

## Definition

"The set A of all points $\underline{x}_0$ of the n-dimensional space is called the <u>region of attraction of self-oscillation</u>, if for $\underline{x}(0) = \underline{x}_0$ belonging to A, it follows that $\rho\left[\underline{x}(t)\right] \longrightarrow 0$ as $t \longrightarrow \infty$ , where $\underline{x}(t)$ is the transient process in (13)."

It is assumed that to every point of the n-dimensional space $\underline{x}_0$ there corresponds a solution of (13), $\underline{x} = \underline{x}(t, \underline{x}_0)$, satisfying the initial conditions $\underline{x}(0) = \underline{x}_0$ . We also assume that $\underline{x}(t, \underline{x}_0)$ is continuous with respect to t and $\underline{x}_0$.

## Theorem 7

"In order that region **A**, consisting of entire trajectories of system (13) and containing the set $\rho(\underline{x}) < \delta$ for sufficiently small $\delta$ , be the region of attraction of self-oscillations of the system (13), it is necessary and sufficient that there exist two functions V and W satisfying the conditions:

1) V is given in **A** and continuous there;

2) W is given in $E_n$ and;

3) V and W vanish at points on the curve $\widetilde{\underline{x}}$;

4) V outside the curve $\widetilde{\underline{x}}$ takes on positive values from (0, +1); the function W is positive and satisfies $W(\underline{x}) > \alpha > 0$ for

$$\rho(\underline{x}) > \beta > 0 \qquad ;$$

5) $\dot{V} = - W(1 - V) \sqrt{1 + f_1^2 + \ldots + f_n^2}$ ;  (15)

6) the function V approaches 1 as $\underline{x}$ approaches $\widetilde{\underline{x}}^*$ where $\widetilde{\underline{x}}^*$ denotes a finite point of the boundary of the region **A**."

## Remark 1

In equation (15) we can replace V by $V_1$ , where $V_1 = - \ln(1 - V)$. As a result of this substitution (15) becomes

$$\dot{V}_1 = - W \sqrt{1 + \sum_{i=1}^{n} f_i^2} \qquad .$$  (16)

Here $V_1$ , in region A, will take on the positive values from $(0, \quad \infty )$, except on the self-oscillations. The function $V_1 \longrightarrow + \infty$ as one approaches a finite or infinite point on the boundary of A. If the function $V_1$ is continuously differentiable with respect to its arguments, then (16) becomes

$$\sum_{i=1}^{n} \frac{\partial V_1}{\partial x_i} f_i = - W \sqrt{1 + \sum_{i=1}^{n} f_i^2} \quad . \tag{17}$$

## Remark 2

If system (13) has self-oscillations, then it may be shown that A is an open and connected set. Also, it may be shown that on the boundary of region A are situated entire trajectories of system (13); that is, if the integral curve of (13) begins on the boundary of the region A, then it remains on it with increasing and decreasing time.

## Remark 3

In the back of this section we consider a third-order system which is simple but at the same time it contains the generality which is characteristic of the behavior of integral curves in nonlinear systems in the presence of self-oscillations.

In reference [1] Zubov applies his partial differential equation technique to obtain Liapunov functions to be used in the analysis of uniformly asymptotically stable trivial solutions of systems of <u>non-stationary differential equations</u>.

Let us consider the following system:

$$\underline{\dot{x}} = \underline{f}(\underline{x}, t), \tag{18}$$

where the right half is specified for $\underline{x}$ belonging to $E_n$ and $- \infty < t < \infty$ and satisfies conditions sufficient to guarantee the existence of a unique

solution for the initial value problem corresponding to (18). The initial

value of $\underline{x}$, $\underline{x}_0$, can be any point in $E_n$ and the initial time satisfies

$-\infty < t_0 < \infty$ . We also assume that the solution of the initial

value problem $\underline{x}(t, \underline{x}_0, t_0)$, depends continuously on $\underline{x}_0$ and $t_0$; and further

we assume that $\underline{x} = \underline{0}$ is a trivial solution of (18); that is, $\underline{f}(\underline{0}, t) = \underline{0}$

for all t.

## Definition

"The trivial solution, $\underline{x} = \underline{0}$, of (18) is <u>stable</u> in the sense of

Liapunov, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that when

$\| \underline{x}_0 \| < \delta$     and $t_0$ belongs to ( $-\infty, \infty$ ), we have

$\| \underline{x}(t, \underline{x}_0, t_0) \| < \epsilon$             for $t > t_0$. If, furthermore

$\| \underline{x}(t, \underline{x}_0, t_0) \| \longrightarrow 0$         as $t \longrightarrow +\infty$     , then $\underline{x} = \underline{0}$ is

<u>asymptotically stable</u>."

## Definition

"If $\underline{x} = \underline{0}$ is asymptotically stable, then the set of all points $(\underline{x}_0, t_0)$

such that     $\| \underline{x}(t, \underline{x}_0, t) \| \longrightarrow 0$         as $t \longrightarrow \infty$     is called the

<u>region A of the asymptotic stability of</u> $(\underline{x}_0 = 0)$."

## Definition

"An asymptotically stable trivial solution of (18) is called <u>uniformly</u>

<u>asymptotically stable</u> if $\| \underline{x}(t, \underline{x}_0, t_0) \| \longrightarrow 0$ as $t - t_0 \longrightarrow \infty$

uniformly with respect to $t_0$ belonging to ( $-\infty, \infty$ ) and $\| \underline{x}_0 \| < \delta$ ,

where $\delta(\epsilon)$ corresponds to the definition of stability."

## Definition

"An asymptotically stable trivial solution of (18) is called <u>uniformly</u>

<u>attracting</u> if for any $h > 0$ , and for $h < \delta$     one can indicate a

$T > 0$ and $\alpha > 0$ such that $\| \underline{x}(t, \underline{x}_0, t_0) \| > \alpha$
when t belongs to $(to, to + T)$        for any $t_0$ in $(-\infty, \infty)$
and for $h \leq \| \underline{x}_0 \| \leq \delta$."

We now list several of <u>Zubov's theorems</u> which are concerned with the
<u>stability of nonautonomous systems</u>. The first two theorems propose certain
differential equations which define V-functions used in the stability
analysis. The third theorem deals with a general system in which a per-
sistent disturbance is present. Zubov's theory shows that his "partial
differential equation" method can be extended to nonautonomous systems
but the actual analytic solution of the differential equations is still
very difficult to obtain.

## Theorem 8

"If any solution of the system (18) is defined for $-\infty < t < \infty$ ,
then in order that a region A, which contains a sufficiently small neighbor-
hood of the set $\underline{x} = \underline{0}$ for all t in $(-\infty, \infty)$, be a region of asymptotic
stability of the uniformly asymptotically stable and uniformly attracting
trivial solution of (18), it is necessary and sufficient that there exist
two functions $V(\underline{x}, t)$ and $\phi(\underline{x}, t)$ having the properties:

(1) V is specified and continuous in A; $\phi$ is specified and continuous
   in $-\infty < t < \infty$    and $E_n$;

(2) $0 < V < + 1$        for $(t, \underline{x})$ in A;    $\phi > 0$    for $\underline{x}$ in $E_n$,
   $-\infty < t < \infty$      if $\| \underline{x} \| \neq 0$ ;

(3) for any $\gamma_2 > 0$ , there exists values $\gamma_1$ and $\alpha_1$ such that $V > \gamma_1$
   for $\| \underline{x} \| \geq \gamma_2$      and $-\infty < t < \infty$    ; $\phi > \alpha_1$
   for $\| \underline{x} \| \geq \gamma_2$      and $-\infty < t < \infty$ ;

(4) $\phi$ and V $\longrightarrow$ 0             uniformly in t as $\|\underline{x}\| \longrightarrow 0$ ;

(5) if $(\underset{\sim}{\underline{x}}, \tilde{t})$ is a point on the boundary of A,   $\|\underline{x}\| \neq 0$    , then

     $\lim V = + 1$    , for $(\underline{x}, t)$ belonging to A, as $\|\underline{x} - \underset{\sim}{\underline{x}}\| \longrightarrow 0$

and $|t - \tilde{t}| \longrightarrow 0$    ;

(6) the total derivative of V, with reference to (18), satisfies the

     equation

$$\dot{V} = - \phi (\underline{x}, t) (1 - V)." \tag{19}$$

Theorem 9

     "In order for a certain region **A**, containing a sufficiently small

vicinity of the axis $\underline{x} = \underline{0}$ and $-\infty < t < \infty$    , to be a region

of asymptotic stability of the trivial solution of (18), it is sufficient,

and in the case

$$\sum_{i=1}^{n} f_i^2 < M \quad \text{for} \quad -\infty < t < \infty, \; \|\underline{x}\| < h, \; M < + \infty, \; h > 0$$

it is also necessary, that there exist two functions V and $\phi$ , having the

properties:

(1) conditions 1 - 5 of Theorem 8 are satisfied;

(2) the total derivative, $\dot{V}$, satisfies

$$\dot{V} = - \phi(1-V) \sqrt{1 + \sum_{i=1}^{n} f_i^2} \quad " \quad . \tag{20}$$

Note

     If it is assumed that $\underline{f}(\underline{x}, t)$, in (18) and $\phi (x, t)$ are differentiable

of a sufficiently high order, then $\dot{V}$ in equations (19) and (20) can be

written as

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(\underline{x}, t) + \frac{\partial V}{\partial t} . \tag{21}$$

Thus, V can be found as a solution of (19) and (21), or (20) and (21), and satisfying the condition $V(\underline{0}, t) = 0$. We also note that if A coincides with the entire space of points $(\underline{x}, t)$, except $x = 0$ and $-\infty < t < \infty$, then the trivial solution of (18) is asymptotically stable in the whole.

In <u>Theorem 10</u>, we examine the system (18) under the influence of a <u>persistent disturbance</u> $\underline{r}(\underline{x}, t)$; that is, we consider the equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) + \underline{r}(\underline{x}, t). \tag{22}$$

The vector function $\underline{r}$ is assumed to satisfy the conditions required for the existence of a unique solution to the initial value problem. If the vector function $\underline{f}$ is continuously differentiable over all the arguments, and if $\underline{x} = \underline{0}$, the trivial solution of (18), is uniformly asymptotically stable, one can show that there exists an upper limit for $\underline{r}(x, t)$ such that when

$$|r_i| < r_* (\underline{x}, t) \qquad \text{the trivial solution of (22) will be}$$

asymptotically stable and will have the same region of asymptotic stability as the trivial solution of (18). <u>Theorem 10</u> deals with the boundedness of the solutions of (22) assuming that the trivial solution of (18) is asymptotically stable.

## Theorem 10

"If the trivial solution (18) is uniformly asymptotically stable, if the condition

$$\sum_{i=1}^{n} f_i^2 < M, \qquad M \equiv \text{constant} \tag{23}$$

is satisfied for t belonging to $(-\infty, \infty)$ and $\|\underline{x}\| < h$, and if there exists asymptotic stability in the whole in system (18), then for a continuously differentiable $\underline{f}(\underline{x}, t)$ in (18) there exists a function $R_1(\underline{x}, t)$ such that for all $r_i(\underline{x}, t)$ any solution of (22) will be <u>bounded</u> when the following is true:

$$|r_i| \leq R_1 (\underline{x}, t) \text{ for } \|\underline{x}\| > H \text{ and } t \text{ in } (-\infty, \infty).\text{''}$$

## WORK OF ZUBOV, MARGOLIS, VOGT, AND RODDEN

In the next series of paragraphs we will consider the application of

Zubov's work to the computation of V-functions and determination of regions

of asymptotic stability through the use of numerical methods and computers.

His method is applied to the partial differential equations (2), (19) and (20).

The form of the V-function used in Zubov's analysis is:

$$V = V_2 + V_3 + \ldots + V_m + \ldots \,, \tag{24}$$

where $V_m$ is homogeneous of degree m in $x_1$, $x_2$, $\ldots$, $x_n$. This power series

for V will converge in a sufficiently small neighborhood of $\underline{x} = \underline{0}$.

The differential equations which are considered by this method of analysis

are:

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \equiv \underline{B}\,\underline{x} + \begin{bmatrix} \displaystyle\sum_{M_1 + \ldots + M_n \geq 2}^{\infty} P_1^{(M_1,\ldots,M_n)} X_1^{M_1} \ldots X_n^{M_n} \\ \\ \displaystyle\sum_{M_1 + \ldots + M_n \geq 2}^{\infty} P_n(M_1, \ldots, M_n) X_1^{M_1} \ldots X_n^{M_n} \end{bmatrix} \tag{25}$$

where $\underline{B}$ is a constant n$^{th}$ order matrix and the P's are constant coefficients.

We assume that the series in (25) are convergent and that $\underline{B}$ is "asymptotically

stable"; that is, the real parts of the eigenvalues of $\underline{B}$ are negative. The

discussion which follows is taken from references [1], [4], [5] and [6].

The following outline of Zubov's work as applied to second order systems

was given in reference [4]. The method of analysis for higher order systems

is similar to the second order case. The system being considered is

$$\dot{x} = f_1(x, y)$$

$$\dot{y} = f_2(x, y). \tag{26}$$

The basic partial differential equation for V is

$$\frac{\partial V}{\partial x} f_1 + \frac{\partial V}{\partial y} f_2 = - \Theta(x, y) \sqrt{1 + f_1^2 + f_2^2} \ (1-V), \tag{27}$$

where $\Theta$ is a positive definite or positive semidefinite form of degree 2M, M $\geqslant$ 1 . Given $\Theta$ , it can be proved that V has a <u>unique</u> power series representation:

$$V(x,y) = V_2(x,y) + \ldots + V_m(x,y) + \ldots$$

$$V(0,0) = 0. \tag{28}$$

Zubov also shows that if the solutions of (26) can be analytically continued for all real t, then (27) can be written in the following form:

$$\frac{\partial V}{\partial x} f_1 + \frac{\partial V}{\partial y} f_2 = - \phi(x, y) (1 - V) \tag{29}$$

where $\phi$ is a quadratic form, (the V's in equations (27) and (29) are obviously different). As in (27), the V satisfying (29) for a given $\phi$ is <u>unique</u> and can be expanded in the power series given in equations (28).

Substituting equation (28) into equation (29) gives the following recursion relationships:

$$\frac{\partial V2}{\partial x} f_{11} + \frac{\partial V2}{\partial y} f_{21} = - \phi_2(x,y) , \tag{30}$$

$$\frac{\partial Vm}{\partial x} f_{11} + \frac{\partial Vm}{\partial y} f_{21} = R_m(x_1y),$$

for m = 3, 4, ...... . In equations (30), the $\phi_2$ function is the quadratic component of

$$\phi(x,y) = - \Theta(x,y) \left\{ 1 + f_1^2 + f_2^2 \right\}^{\frac{1}{2}} =$$

$$= \phi_2(x,y) + \phi_3(x,y) + \ldots + \phi_m(x,y) + \ldots , \tag{31}$$

where $\phi_m$ is a homogeneous form of $m^{th}$ power in x and y; the $V_2(x, y)$ is a positive definite form defined in (28); $f_{11}$ and $f_{21}$ are given by

$$f_{11} = a_{11} x + a_{12}y ,$$

$$f_{21} = a_{21} x + a_{22}y ,$$

(32)

which are the linear terms in $f_1$ and $f_2$ of (26); $R_m$ is a known function of the previously computed $V_2$, $V_3$, ... , $V_{m-1}$ functions; and the defining equation for $R_m$, as given in [6] , is

$$R_m = - \phi_m + \sum_{j+k=m} \phi_j V_k +$$

$$- \sum_{j+K=m+1} \left\{ \sum_{m_1 + m_2 = j} P_1 (m_1, m_2) x^{m_1} y^{m_2} \frac{\partial}{\partial x} + \right.$$

(33)

$$\left. + \sum_{m_1 + m_2 = j} P_2(m_1, m_2) x^{m1} y^{m2} \frac{\partial}{\partial y} \right\} V_k ,$$

where j, k, m = 2, 3, 4, ... and the P's are given by equation (25).

Equation (29) can be transformed into the following if V is replaced by

$$V_* = - \ln(1 - V) ;$$

that is, (29) becomes

(34)

$$\frac{\partial V*}{\partial x} f_1 + \frac{\partial V*}{\partial y} f_2 = - \phi (x,y).$$

(35)

When $0 \leq V < 1$, $V_*$ satisfies $0 \leq V_* < \infty$ .

In the list of theorems, which are concerned with the application of <u>Zubov's Approximate Methods</u> in the analysis of the stability of second order systems, the following definitions are required.

Definition 1

"Region A is defined to be the <u>domain of asymptotic stability</u> of system (26)."

## Definition 2

"G($\lambda$) is the set of points (x, y) in the phase plane which satisfy the inequality $0 \leq V(x, y) < \lambda$ for any $\lambda$ belonging to (0, 1)."

## Definition 3 (Modified in [6] )

"The set $W_2$ consists of all points of zero $\dot{V}_2$ which define boundaries between regions of positive and negative $\dot{V}_2$, while excluding from $W_2$ those points of zero $\dot{V}_2$ which lie in surrounding regions of $\dot{V}_2$ with constant sign, $\dot{V}_2$ being of constant sign except for these exceptional points. The set $W_1$ contains those exceptional points. Designate by $C_1$ the smallest value of $V_2$ on set $W_2$, and $C_2$ as the largest value of $V_2$ on $W_2$. (The reason for the existence of set $W_1$ is that it is sometimes convenient to use semidefinite $\phi$'s in (29) and (35).)"

## Definition 4

"The <u>region A is bounded</u> if there exists a positive constant R such that $x^2 + y^2 = R$ encloses A."

## Definition 5

"Let $V^{(n)}$ (x, y) equal the finite sum, $\sum\limits_{i-2}^{n} V_i (x, y)$ . Then as in [6], we define $W_2^{(n)}$ (x, y) as all the points (x, y) on which $\dot{V}^{(n)}$ (x, y) = 0, other than those (set $W_1^{(n)}$ ) for which

$$\dot{V}^{(n)} (x + \delta x, y + \delta y) \leq 0$$

or

$$\dot{V}^{(n)} (x + \delta x, y + \delta y) \geq 0$$

for all $\delta_x$ and $\delta_y$ infinitesimally small.

Let $C_1^{(n)}$ be the minimum of $V^{(n)}$ on $W_2^{(n)}$, and let $A^{(n)}$ be all (x, y) such $V^{(n)} \leq C_1^{(n)}$ ."

We now give a <u>brief outline</u> in "theorem form", of <u>Zubov's Construction</u> <u>Technique</u> for obtaining approximations to the domain of asymptotic stability.

## Theorem 1

"If (x, y) belongs to region A of definition 1, then the V-function defined by equation (29) satisfies the inequality $0 \leq V < 1$. Thus, we also have $V_*$ of equation (35) satisfying $0 \leq V_* < \infty$."

## Theorem 2

"If $(\xi, \eta)$ is a boundary point of A and (x, y) belongs to A, then then $\lim_{(x,y) \to (\xi, \eta)} V(x, y) = 1$ , or $\lim_{(x, y) \to (\xi, \eta)} V^*(x,y) = \infty$."

## Theorem 3

"If $\lambda$ belongs to (0, 1), then set $G(\lambda)$ is a bounded domain in domain A."

## Theorem 4

"The curve $V = 1$ is an integral curve of system (26)."

## Theorem 5

"If $\phi$ is given, then the solution, V, of (29) is unique in domain A."

## Theorem 6

"The boundary of A is a family of curves defined by $V = 1$."

## Theorem 7

"If x = 0, y = 0 is asymptotically stable in the whole, then $V < 1$ for all (x, y) in the phase plane."

## Theorem 8

"Let V be the solution of (29). Then $V = \lambda$, $\lambda$ equal a constant, is the solution of the system $\dot{x} = \dfrac{\partial V}{\partial y}$ and $\dot{y} = -\dfrac{\partial V}{\partial x}$

such that if $\lambda_1 < \lambda_2$ , then $V = \lambda_1$ is enclosed inside
$V = \lambda_2$."

## Theorem 9

"For any system $\overset{\bullet}{x} = f_1 (x, y)$ and $\overset{\bullet}{y} = f_2(x, y)$ there can be related an

entire class of systems of the form $\overset{\bullet}{x} = \dfrac{\partial V}{\partial y}, \overset{\bullet}{y} = - \dfrac{\partial V}{\partial x}$

where V depends upon the form of $\phi$ . (From the previous theorem each closed

curve $V = \lambda_1$ has no parts in common with $V = \lambda_2$ if $\lambda_1 \neq \lambda_2$ .

Thus, from this result the next conclusion follows.) The boundary of domain A

will be the only common integral curve of the two systems for $0 < \lambda \leq 1$ ."

Theorem 9 is a very important theorem in obtaining approximate domains

of asymptotic stability. In references [4] and [5], Theorem 9 and the

"approximation" theorems given below are applied to Van-der-Pol's equation,

other second-order equations, and a third-order equation.

## Theorem 10

"Let L be an integral curve of (26) which lies on the boundary of A. Then

there exists a value $C_4$ such that $V_2 = C_4$ is a curve which is tangent to L at

$(x_0, y_0)$."

## Theorem 11

"The curve $V_2 = C_1$ , of definition 3, is contained in A provided that the

set $W_1$, for $V_2 < C_1$, is not a half-trajectory of the system."

## Theorem 12

"If A is bounded, $V_2 = C_1$ is bounded for any permissible $\phi$ ."

## Theorem 13

"If $V_2 = C_1$ is unbounded, then **A** is unbounded."

## Theorem 14

"If $\phi$ in (29) is an admissible function and $C_2$ is finite, then domain **A** is bounded and its boundary lies in the region $C_1 \leqq V_2 \leqq C_2$."

The **next theorems** are concerned with the higher order approximations for V. As will be pointed out later, these higher order terms need not give better approximations to domain **A** than $V_2$.

Thus, the application of Theorems 12, 13 and 14 may give fairly good results and the higher order terms need not be computed.

## Theorem 15

"The curve $V^{(n)} = C_1^{(n)}$ , n = 2, 3, ... , is contained in **A** provided that the set of points $W_1^{(n)}$ for $V < C_1^{(n)}$ is not a half-trajectory of the system."

## Theorem 16

"If **A** is bounded, then $V^{(n)} = C_1^{(n)}$ is bounded for any admissible $\phi$ . If any $V^{(n)} = C_1^{(n)}$ is unbounded, then **A** is unbounded."

Thus, from $V^{(n)} = C_1^{(n)}$ , we can approximate the region of asymptotic stability. As $n \rightarrow \infty$ , $C_1^{(n)} \longrightarrow$ 1. In references [4] and [5] second and third order examples were investigated by making use of the above theorems and electronic computers. In Rodden's work, [5], equation (29) was called the "regular" equation and (35) was called the "modified" equation. We will now discuss the results of these studies.

Both authors, [4] and [5], studied Van-der-Pol's equation,

$$\ddot{x}_1 + \epsilon (1 - x_1^2) \dot{x}_1 + x_1 = 0, \quad x_2 = \dot{x}_1 + \epsilon (x_1 - x_{1/3}^3). \quad \text{For} \quad \phi = x_1^2 \quad \text{in}$$

(29) both authors obtained similar results. That is, $V^{(4)}$ gave the worst approximation for A, $V^{(6)}$ a better approximation, and then $V^{(2)}$, $V^{(10)}$, $V^{(14)}$, and $V^{(20)}$, in that order of increasing improvement of the approximate stability domain, where $V^{(20)}$ was very close to the actual domain A. In [5], Rodden replaced $\phi = x_1^2$ by $\phi = x_1^2 + x_2^2$. The results in the order of best approximation (first V giving the worst approximation) are $V^{(6)}$, $V^{(2)}$, $V^{(10)} \approx$ $\approx V^{(2)}$, $V^{(18)} \approx V^{(20)}$. But for this case, the boundary defined by $V^{(20)}$ is not at all close to the actual boundary of A. The conclusions which can be made from the above analysis are: (1) the convergence of the series in (28) is not uniform, (2) the convergence of series (28) may be very slow, and (3) the choice of $\phi$ in (29) and (35) greatly influences the accuracy of the approximation to the domain of asymptotic stability, A. An example from Zubov, [1], shows why conclusion (3) may often be valid. Let partial differential equation for V be given as:

$$\sum_{i=1}^{n} x_i \frac{\partial V}{\partial x_i} = \phi(\underline{x})(1 - V) . \tag{36}$$

If $\phi(\underline{x}) = 2 \sum_{1}^{n} x_i^2$, the solution of (36) is

$$V(\underline{x}) = 1 - \exp\left\{ -\sum_{i-1}^{n} x_i^2 \right\} . \tag{37}$$

If $\phi(\underline{x}) = (1 + \sum_{i=1}^{n} x_i^2)^{-3/2} (\sum_{i=1}^{n} x_i^2)$, the solution of (36) is

$$V(\underline{x}) = 1 - \exp\left\{ (1 + \sum_{i=1}^{n} x_i^2)^{-\frac{1}{2}} - 1 \right\} . \tag{38}$$

If we seek $V(\underline{x})$ in the form of a series solution, (28), equation (37) would

have a series expansion which would converge in the whole space, while the series corresponding to (38) would converge in a <u>bounded</u> part of the space.

In reference (4), Margolis concludes by stating that he is attempting to program the Zubov stability analysis for the second order problem

$$\ddot{x} + c(x) \dot{x} + r(x) x = 0, \tag{39}$$

where c and r are polynomials or convergent power series in x, having non-zero constant terms. He also mentions the need for further research on computer programs for higher order systems and for nonanalytic nonlinearities.

To conclude the discussion of this work let us consider some more of Rodden's numerical results, [5]. He considered the second order example

$$\dot{x} = y$$

$$\dot{y} = -x - y + x^3 .$$

For the "regular" Zubov equation and with a semidefinite $\phi$, $\phi = x^2$, he found a lack of uniform convergence to the domain of stability of the sequence $\left\{ v^{(n)} \right\}$. When he used the "modified" Zubov equation and the same $\phi$, the convergence became more uniform but the rate of convergence to the domain of asymptotic stability was slower. Rodden also studied the second order system given by

$$\dot{x} = x + 2x^2 y$$

$$\dot{y} = y.$$

He applied both the regular and modified Zubov equations with $\phi = 2(x^2 + y^2)$ and obtained the same results as in the previous case; that is, nonuniform convergence and a slow rate of convergence of the $v^{(n)}$'s.

Lastly, Rodden considered a third order system defined by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -b_1 x_3 - (b_2 + c_2 b_3) x_2 - b_3 x_1 - b_4 (x_1 + c_2 x_2)^3 .$$

In this system the coefficients are positive and it can be shown that a sufficient condition for global stability of the system is that $b_1 C_2 > 1$, where $C_2$ is called the "damping term". Let $C_{2c}$ be the critical value of $C_2$; that is, $b_1 C_{2c} = 1$. Thus, for $C_2$'s satisfying $0 < C_2 < C_{2c}$, the region of stability of the system is __not__ global. In Rodden's example, $C_2$ was equal to $C_{2c}/2$. He determined $V^{(2)}$, $V^{(4)}$ and $V^{(6)}$ from both the regular and modified Zubov equations for a semidefinite $\phi = x_1^2 + x_2^2$. His conclusions were the same as in the previous systems; that is, the convergence appears to be nonuniform and the rate of convergence is slow.

## SZEGÖ'S WORK

In reference [9], Szegö investigates the stability properties of the solution $\underline{x} = \underline{0}$ of nonlinear autonomous systems by considering the partial differential equation suggested by Zubov. He proves that this problem can be reduced to the construction of a matrix $\underline{A}(\underline{x})$ whose elements have the form $a_{ij} \equiv a_{ij}(x_i, x_j)$. The same result was reached in a different way in one of Szegö's earlier papers in which the nonlinearities occurred in polynomial form. The discussion of this earlier work is considered in another section of this report. Szegö claims that the work in [9] is not limited to the investigation of only one singular point but can be applied to systems which have several isolated singular points.

In references [10,11] Szegö introduces a new partial differential equation for the stability analysis of autonomous control systems. This new equation turns out to be a generalization of the partial differential equations of Zubov.

We will first discuss the work of reference [9]. The system being considered is

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \ , \ \underline{f}(\underline{0}) = \underline{0} \ , \tag{40}$$

where the usual conditions for existence, uniqueness and continuity of solutions are satisfied. We seek a function $V(\underline{x})$ such that

$$\dot{V} = (\underline{\nabla V})_T \, \underline{f} \, (\underline{x}) = \gamma \, (\underline{x}) \,, \tag{41}$$

where $\gamma \, (\underline{x})$ is any scalar function which satisfies either the condition of definiteness on the trajectories of (40) or has the form

$$\gamma (\underline{x}) = \Theta (\underline{x}) \, g \left[ \xi (\underline{x}) \right]. \tag{42}$$

In (42), $\Theta \, (\underline{x})$ is definite on the trajectories of the system (40) and $g(\underline{x})$ is indefinite on a closed surface, where we define $g \left[ \xi (\underline{x}) \right]$ to be indefinite on a closed surface if $g(0) = 0$ and $g(u)/u > 0$ . (The equation $\xi (\underline{x}) = 0$ defines a closed, bounded surface.)

The case in which the system is locally stable, equation (42), has been previously studied by the author and is discussed in another section of this report. In this section, we will discuss the case in which $\gamma \, (\underline{x})$ is definite on the trajectories of system (40). For stability, not asymptotic stability, $\gamma \, (\underline{x})$ may be zero on the trajectories of the system. Thus, equation (41) is a partial differential equation analogous to those of Zubov.

Szegö assumes that $(\underline{\nabla V})_T$ in (41) can be replaced by $\underline{x}_T \, \underline{A}(\underline{x})$ where the elements of $\underline{A}(\underline{x})$ are of the form

$$a_{ij} \equiv a_{ij} \, (x_i, \, x_j). \tag{43}$$

Szegö invokes the theorem which states that "a necessary and a locally sufficient condition for $\underline{x}_T \, \underline{A}(\underline{x})$ to satisfy the equation

$$\underline{x}_T \, \underline{A}(\underline{x}) = (\underline{\nabla V})_T \tag{44}$$

is that the matrix

$$\underline{D}(\underline{x}) = \left\{ \frac{\partial (\underline{x}_T \, \underline{A}(\underline{x}))_i}{\partial \, x_j} \right\} \tag{45}$$

be symmetric." The symmetry of $\underline{D}(\underline{x})$ places the following restrictions on $\underline{A}(\underline{x})$:

$$a_{ij}(x_i,x_j) + x_i \frac{\partial a_{ij}(x_i,x_j)}{\partial x_i} = a_{ji}(x_i,x_j) + x_j \frac{\partial a_{ji}(x_i,x_j)}{\partial x_j} . \qquad (46)$$

From equations (41), (43), (44), (45) and (46), the gradient of V can be determined. By the usual line integration techniques we can obtain $V(\underline{x})$; for example, in the second order system V is given by

$$V(\underline{x}) = \int \left\{ x_1 a_{11}(x_1) + x_2 a_{12}(x_1,x_2) \right\} dx_1 + \xi_1(x_2) , \qquad (47)$$

or

$$V(\underline{x}) = \int \left\{ x_1 a_{21}(x_1,x_2) + x_2 a_{22}(x_2) \right\} dx_2 + \xi_2(x_1),$$

where $\xi_1$ and $\xi_2$ are arbitrary functions. It is possible that for some third some third and fourth order systems the problem may have to be formulated in such a way that more than one unknown matrix must be determined. This problem is briefly discussed in [9].

In order to construct the matrix $\underline{A}(\underline{x})$ which satisfies $\gamma(\underline{x}) = \underline{x}_T \underline{A}(\underline{x}) \underline{f}(\underline{x})$, we consider the equation:

$$\gamma(\underline{x}) = \underline{x}_T \underline{A}(\underline{x}) \underline{f}(\underline{x}) = 0. \qquad (48)$$

The solutions of (48) define surfaces in the Euclidean space. In order that $\gamma(\underline{x})$ be definite on the trajectories of the system, it must not change sign across these surfaces. Assuming continuity of $\gamma(\underline{x})$, these surfaces correspond to roots of even order of multiplicity of the equation (48).

Therefore, to construct the matrix $\underline{A}(\underline{x})$ we solve (48) with respect to one of the components of $\underline{x}$, say $x_i$, and require that real roots of this equation have even order of multiplicity. Geometrically speaking, we require that the

matrix $\underline{A}(\underline{x})$ to be such that even numbers of surfaces, solutions of (48), coincide. This fact is illustrated for a third order system in the compendium of examples. A concluding remark is that the above method for generating Liapunov functions is very similar to the Variable Gradient Method.

We now will discuss Szegö's work which is presented in references [10] and [11]. This study is limited to the investigation of completely defined systems; that is, given a control system we wish to determine the stability properties of the equilibrium point. (This, of course, is really our concern in this whole report.) Szegö claims that the method presented below will work as long as the Liapunov functions belong to the class of functions which are solutions of a generalized Zubov partial differential equation.

Given the nonlinear autonomous dynamic system in equation (40), the problem of stability analysis of $\underline{x} = \underline{0}$ is then formally reduced to the search for a positive definite scalar function $\gamma(\underline{x})$ and a scalar function $v = v(\underline{x})$ , $v(\underline{0}) = 0$, such that the partial differential equation

$$\dot{v} = (\underline{\nabla v})_T \underline{f}(\underline{x}) = - \gamma(\underline{x}) \tag{49}$$

is satisfied. The inverse stability theorems guarantee that such scalar functions $\gamma(\underline{x})$ and $v(\underline{x})$ exist. But, it is not always practical to search for a positive definite $\gamma(\underline{x})$. Therefore, a more sensible approach is that of finding a sufficient condition which guarantees the existence of a $\gamma(\underline{x})$ which is at least positive definite on the trajectories of (40). Szegö's method gives a procedure to determine $\gamma$ and v such that (49) is satisfied. In the following paragraphs we discuss the "highlights" of Szegö's method as presented in reference [10].

First, we consider a scalar function $v_1(\underline{x})$, $v_1(\underline{0}) = 0$ ; its time derivative with respect to equation (40) is

$$\dot{v}_1 = (\nabla v_1)_T \, \underline{f}(\underline{x}) = \gamma(\underline{x}) \, , \quad \gamma(\underline{0}) = 0. \tag{50}$$

Next we look for a $\psi(\underline{x})$ which is definite on the trajectories of (40) and

a scalar function $\beta(v_1)$, $\quad \int_0^{v_1} \beta(s) \, ds < \infty \quad$ , such that

$$\frac{\psi(\underline{x})}{\gamma(\underline{x})} = \beta(v_1) \, . \tag{51}$$

Thus, the differential equation

$$\frac{d\alpha(v_1)}{d v_1} = \frac{\psi(\underline{x})}{\gamma(\underline{x})} = \beta(v_1) \tag{52}$$

can be integrated. The solution of (52) is given as $\alpha \equiv \alpha(v_1) \equiv \alpha^*(\underline{x})$

and the corresponding time derivative is

$$\dot{\alpha}^*(\underline{x}) = (\nabla \alpha^*(\underline{x}))_T \underline{f}(\underline{x}) = \frac{d\alpha}{dv_1} (\nabla v_1)_T \, \underline{f}(\underline{x})$$

$$= \frac{d\alpha}{dv_1} \gamma(\underline{x}) = \psi(\underline{x}). \tag{53}$$

Therefore, $\dot{\alpha}^*(\underline{x})$ is definite on the trajectories of (40); and because of

the assumptions made about $\alpha(v_1)$ and $v_1(\underline{x})$ , $\alpha^*$ solves the stability

problem.

Combining equations (50) and (51) gives us the following generalized

Zubov equation, compare (54) with equation (19):

$$\dot{v}_1 \equiv (\nabla v_1)_T \, \underline{f}(\underline{x}) = \frac{\psi(\underline{x})}{\beta(v_1)} \, . \tag{54}$$

The stability results obtained from (54) are summarized in Theorems 1 and 2.

### Theorem 1

"The stability theory of $\underline{x} = \underline{0}$ of (40) is reduced to finding the scalar

functions $v_1(\underline{x})$, $\psi(\underline{x})$ and $\beta(v_1)$ such that $v_1(\underline{0}) = 0$ , $\int_0^{v_1} \beta(s) \, ds < \infty$

and $\psi(\underline{x})$ is definite on the trajectories of (40)."

## Theorem 2

"The solution $\underline{x} = \underline{0}$ of (40) is <u>asymptotically stable</u> in a closed, bounded region $S, \overset{*}{\alpha}(\underline{x}) \leq \delta$ , $\delta > 0$, if there exist scalar functions $v_1(\underline{x}), \gamma(\underline{x})$ and $\beta(v_1)$ satisfying the conditions:

(i)   $v_i(0) = 0$

(ii)  $\gamma(\underline{x})$ is negative definite on the trajectories of (40),

(iii) $\displaystyle\int_0^{v_1} \beta(s)\,ds = \alpha(v_1) < \infty$

(iv)  $\overset{*}{\alpha}(\underline{x}) = \alpha(v_1) = \displaystyle\int_0^{v_1} \beta(s)\,ds > 0$ in $S,\ \underline{x} \neq \underline{0}$ ,

$\overset{*}{\alpha}(\underline{o}) = 0$

and

(v)   equation (54) is satisfied."

## Corollary 1

"The solution $\underline{x} = \underline{0}$ of (40) is <u>asymptotically stable in the large</u> if all the conditions of Theorem 2 are satisfied and

$$\lim_{\|\underline{x}\| \to \infty} \overset{*}{\alpha}(\underline{x}) = \lim_{\|\underline{x}\| \to \infty} \int_0^{v_1} \beta(s)\,ds = \infty."$$

## Corollary 2

"If all the conditions of Theorem 2 are satisfied with the sign of $\gamma(\underline{x})$ changed, then $\underline{x} = \underline{0}$ of (40) is <u>completely unstable.</u>"

We now consider a simplified procedure for constructing Liapunov functions.
We seek a scalar function $v_2 = v_2(\underline{x})$, $v_2(\underline{o}) = 0$, such that

$$\dot{v}_2 = (\nabla v_2)_T \; \underline{f}(\underline{x}) = \Theta(v_2), \tag{55}$$

where $\Theta = \Theta(v_2)$ is a bounded scalar function. Thus, we can find a
semidefinite $\mathcal{J}(v_2(\underline{x})) = \overset{*}{\mathcal{J}}(\underline{x})$ such that

$$\frac{d\alpha_2(v_2)}{dv_2} = \frac{\mathcal{J}(v_2)}{\Theta(v_2)} = \beta_2(v_2), \tag{56}$$

is integrable. The solution of (56) is $\alpha_2(v_2) = \overset{*}{\alpha}2(\underline{x})$
and the corresponding time derivative is

$$\dot{\alpha}_2 = (\nabla\overset{*}{\alpha}2)_T \; \underline{f}(\underline{x}) = \overset{*}{\mathcal{J}}(\underline{x}), \tag{57}$$

where $\overset{*}{\mathcal{J}}$ is semidefinite. If no degeneracy occurs, $\overset{*}{\alpha}2 = 0$
when $\underline{x} \neq \underline{0}$, and if

$$\int_0^{v_2} \beta_2(S) \, dS < \infty, \tag{58}$$

then $\overset{*}{\alpha}_2(\underline{x})$ is a Liapunov function of (40). Szego proves the
following existence theorem for the solutions of (55), a similar one is
possible for (54).

## Theorem 3

"There always exists a scalar function $\Theta(v_2)$ such that (55) has a solution which satisfies $v_2(\underline{o}) = 0$. In particular if (40) is asymptotically
stable, $v_2(\underline{x})$ is definite and $\Theta(v_2)$ may be chosen so that

$$\Theta(v_2) = \lambda v_2, \qquad \text{Re}(\lambda) < 0 \qquad \text{''} .$$

## Remark

The major difference between (54) and (55) is that the Liapunov function
derived from (54) is never degenerate while the one from (55) may be
degenerate.

In conclusion, we observe that the essence of Szegö's method is the introduction of the functions $\beta(v_1)$ and $\mathcal{Q}(v_2)$ respectively in (54) and (55). The important consideration is to find a scalar function $v = v(\underline{x})$, $v(\underline{0}) = 0$, such that $\dot{v}$ has the form $\mathcal{Q}_2(v)$ or $\gamma(\underline{x})/\beta(v)$, but otherwise is arbitrary. Thus, we are constructing a Liapunov function by solving a quasi-linear partial differential equation, (54) and (55), whose right hand side has a well-defined form.

## Change of Variable

Szegö considers a change of variable, $\underline{Z}$ for $\underline{x}$, such that one component of $\underline{Z}$ is the scalar function $v$. The aim of the transformation is that by using the well-defined form of the right sides of (54) and (55) the stability problem can be reduced to a search for a scalar function $\xi = \xi(\underline{Z})$, satisfying satisfying a certain nonlinear partial differential equation. The right hand side of this partial differential equation is any definite function which depends on only one component of $\underline{Z}$.

The results of this transformation applied to (54) and (55) are respectively (59) and (60):

$$\left\{ f_i(\underline{x}) - \sum_{j=1}^{n} \left[ \frac{\partial \xi_i}{\partial z_j} f_j(\underline{x}) \right] \right\}_{x_i = \xi_i} = \left. \frac{\gamma(\underline{x})}{\beta(x_1)} \right| \cdot \left. \frac{\partial \xi_i}{\partial w_i} \right|_{x_i = \xi_i}, i \neq j \qquad (59)$$

$$\left\{ f_i(\underline{x}) - \sum_{j=1}^{n} \frac{\partial \xi_{i*}}{\partial z_j} f_j(\underline{x}) \right\}_{x_i = \xi_i} \cdot \frac{1}{\partial \xi_{i*}/\partial w_2} = \mathcal{Q}(w_2), j \neq 1 \qquad (60)$$

where $Z_K = x_K$, $K \neq i$ and $Z_i = w_i$.

Therefore, the only requirement we have is that the right-hand side of (60) depends on $w_\lambda$. Whatever the function $\Theta(w_2)$ is, we always are able to give some information about the stability of the system; that is, we can either determine the stability of $\underline{x} = \underline{0}$, or we can determine the stability of some first integral of the system going through the origin. Usually the solution of the above partial differential equation is difficult; therefore, a more reasonable approach is to choose $\xi_i^* = \xi_i^* (\underline{x})$, $\xi_i^* (\underline{0}) = 0$ , with unknown coefficients. Then the unknown coefficients are computed in such a way that the right-side of (60) is a function of $w_2$ only. Szegö gives no general method to get $\xi_i^*$. An example of this problem is given at the end of this section.

In reference [11], Szegö discusses some <u>extensions of his work in [10]</u> by considering some of the structural aspects of the stability investigation of system (40). The results of this work are summarized in the following theorem which gives the stability properties of a manifold, M, in Euclidean space.

## Theorem 4

"Consider the dynamical system given by equation (40). If

(i)     $v(\underline{x})$ is a continuous scalar function with continuous first partials in the whole space $E_n$,

(ii)    $\Theta(v)$ is a continuous scalar function,

(iii)   M is a manifold on which $v(\underline{x}) = 0$,

(iv)    $\Theta(v(\underline{x})) \equiv 0$ at all points of M,  $\Theta \neq 0$  for $\underline{x}$ not in M,

(v)     the equation

$$(\nabla v(\underline{x}))_T \, \underline{f}(\underline{x}) = - \Theta(v)$$

is satisfied in $E_n$,

(vi)   $v(\underline{x}) \, \Theta(v(\underline{x})) \geq 0$ in $E_n$,

(vii)   the trivial solution, $v = 0$, of $\dot{v} = -\Theta(v)$ is globally asymptotically

    stable,

(viii) (I)   $a(\rho(\underline{x}, M)) \leq |v(x)|$

     (II)   $a(\rho(\underline{x}, M)) \leq |v(\underline{x})| \leq b(\rho(\underline{x}, M))$ ,

where $\rho(\underline{x}, M)$ is the Euclidean distance of the point $\underline{x}$ from the set $M$,

$a(r)$ and $b(r)$ are positive definite scalar functions, and $a(r)$ satisfies

$$\lim_{r \to \infty} a(r) = \infty \quad ,$$

then, if (I) is satisfied, $\lim_{t \to \infty} \rho(\underline{x}(t), M) = 0$ for all initial

conditions; and if (II) is satisfied, M is globally asymptotically stable." 

## Remarks

We conclude this section with a few remarks about the special cases of the

above theorem.   (1) If M is a <u>minimal</u> set containing the equilibrium

point $\underline{x} = \underline{0}$, asymptotic stability of M implies asymptotic stability of $\underline{x} = \underline{0}$.

(2) If $n = 2$ and M is a closed, bounded curve not containing $\underline{x} = \underline{0}$, M corre-

sponds to a periodic motion.  If M is unbounded, then either all its points

are equilibrium points or M corresponds to a singular solution.   (3) Similar

results are obtained if $n = 3$.   (4) The stability problem of (40) is reduced

to the identification of M, and thus reduced to a problem of dimension of

at most $n-1$, the dimension of M.   (5) Examples of this work are given in the

compendium.

## COMPENDIUM OF EXAMPLES

**Example 1, [1]**    **Second Order Case**

The defining equations of the system are

$$\dot{x} = -x + 2x^2 y \,,$$

$$\dot{y} = -y.$$

The partial differential equation of Zubov which defines the V-function is

$$\frac{\partial V}{\partial x} (-x + 2x^2 y) + \frac{\partial V}{\partial y} (-y) = -\phi (\underline{x}) (1 - V)$$

where $\phi$ is taken to be $\phi = x^2 + y^2$. By direct substitution, one can verify that a solution of this equation is given by the following:

$$V(x,y) = 1 - \exp\left\{ -y^2/2 - \frac{x^2}{2(1 - xy)} \right\}.$$

As $xy \longrightarrow 1^-$, $V \longrightarrow 1^-$. Thus, the curve $xy = 1$ forms the boundary of the region of stability about $(0, 0)$. That is, for every initial point $(x_0, y_0)$ such such that $x_0 y_0 < 1$, the subsequent motion defined by the above system approaches $(0, 0)$ as $t \rightarrow \infty$.

**Example 2, [1]**    **Second Order Case**

The system is defined by

$$\dot{x} = f_1 (x,y) = -3x + 3a(x/a)^{2/3} (y/b)^{1/3} + 3a(x/a)^{5/3} + 3x (y/b)^{2/3}$$

$$\dot{y} = f_2(x,y) = -3y + 3b (y/b)^{2/3} (x/a)^{1/3} + 3b(y/b)^{5/3} + 3y(x/a)^{2/3}$$

The V-function is obtained from the following partial differential equation:

$$\frac{\partial V}{\partial x} f_1 (x,y) + \frac{\partial V}{\partial y} f_2 (x,y) = -2 \mu \left\{ (x/a)^{2/3} + (y/b)^{2/3} \right\} (1-V)$$

where $\mu > 0$. The continuous solution of this equation satisfying

$V(0,0) = 0$ is given by

$$V(x,y) = 1 - \left\{ 1 - (x/a)^{2/3} - (y/b)^{2/3} \right\}^{M},$$

from which it follows that the integral curve, bounding the region of stability, is defined by

$$(x/a)^{2/3} + (y/b)^{2/3} = 1.$$

The V-function in this case is not everywhere differentiable, but $\overset{\bullet}{V}$ is continuous in the region of asymptotic stability. The family of sections of the region of asymptotic stability is given by

$$\left\{ 1 - (x/a)^{2/3} - (y/b)^{2/3} \right\}^{M} = \lambda \; ; \quad 0 < \lambda < 1.$$

### Example 3. [1]    Second Order Case

The following system has a rest point at $(1, 0)$:

$$\overset{\bullet}{x} = \frac{(1 - x^2 + y^2)2x}{(x + 1)^2 + y^2} + xy,$$

$$\overset{\bullet}{y} = \frac{1 - x^2 + y^2}{2} - \frac{4yx^2}{(x+1)2 + y^2}.$$

The corresponding partial differential equation has the form

$$\frac{\partial V}{\partial x} \left\{ \frac{(1 - x^2 + y^2)\, 2x}{(x+1)^2 + y^2} + xy \right\} +$$

$$+ \frac{\partial V}{\partial y} \left\{ \frac{1 - x^2 + y^2}{2} - \frac{4yx^2}{(x+1)^2 + y^2} \right\} = - 2 \frac{(x-1)^2 + y^2}{(x+1)^2 + y^2} (1 - V).$$

By substitution, one can verify that

$$V = \left\{ \frac{(x-1)^2 + y^2}{(x+1)^2 + y^2} \right\}$$

is a continuous function which satisfies the above partial differential equation and the condition $V(1, 0) = 0$. From $V = 1$, we obtain the boundary

of the region of asymptotic stability for the rest point $(1, 0)$. This boundary

is the y-axis and the region of asymptotic stability is the right half plane.

The equation for the family of cross sections which fill the region of asymptotic

stability are obtained from the equation

$$V(x,y) = 1 - \lambda , \quad 0 < \lambda < 1 .$$

The sections are circular, defined by:

$$\left\{ x + 1 + \frac{-2}{\lambda} \right\}^2 + y^2 = \left\{ 1 - 2/\lambda \right\}^2 - 1.$$

### Example 4, [1]    Second Order System

In this example, we consider the system

$$\dot{x} = \delta(x) + \sigma(y) ,$$

$$\dot{y} = \Theta(x) ,$$

where the right sides of the equations are specified for any x and y and

satisfy the sufficient conditions that guarantee the existence of the unique

solution $x (t, x_o, y_o)$ and $y (t, x_o, y_o)$                for all

finite values $x_0$ and $y_0$, and having a unique rest point at $x = y = 0$. We also

assume that $\delta(x)$ and $\Theta(x)$ have signs opposite to the signs of x, and that

$\sigma(y)$ has the same sign as y.

The defining partial differential equation for V is given by

$$\frac{\partial V}{\partial x} (\delta + \sigma) + \frac{\partial V}{\partial y} (\Theta) = - \delta \Theta (1-V).$$

In this case the $\phi(x,y)$ function which is used is given by

$$\phi = \delta(x) \Theta(x) ,$$ from which it follows that $\phi(0,y) = 0$        for

any y. It can be shown that Zubov's theory still is valid for this choice

of $\phi$ .

The function defined by

$$V(x,y) = 1 - \exp\left\{\int_0^x \Theta(\tau)\,d\tau - \int_0^y \sigma(\tau)\,d\tau\right\}$$

is the solution of the above partial differential equation. If the integrals, $\int_0^x \Theta(\tau)\,d\tau$ and $\int_0^y \sigma(\tau)\,d\tau$ tend to infinity as

$$|x| \longrightarrow \infty \quad \text{and} \quad |y| \longrightarrow \infty$$ , then (0, 0) is asymptotically stable over the entire space.

## Example 5, [1]  $n^{\underline{th}}$ Order System

The following $n^{\underline{th}}$ order system is an example of Theorem 3, under Zubov's work:

$$\dot{\underline{x}} = \underline{P}\,\underline{x} + \underline{f}(\underline{x}) ,$$

where $\underline{P}$ is a constant n-by-n matrix whose eigenvalues all have negative real parts. The problem is to find out for which functions $\underline{f}(\underline{x})$ the trivial solution $\underline{x} = \underline{0}$ of the above system is asymptotically stable on the whole.

First, for the linear system $\dot{\underline{x}} = \underline{P}\,\underline{x}$ we construct the following V - function, given as a positive-definite quadratic form

$$V = \underline{x}_T\,\underline{A}\,\underline{x}, \qquad\qquad \underline{A} \equiv \text{constant},$$

and

$$\dot{V} = W(\underline{x}) = \underline{x}_T\,\underline{B}\,\underline{x} ,$$

$$\underline{B} = \underline{P}_T\,\underline{A} + \underline{P}\,\underline{A} ,$$

where $\underline{B}$ is negative definite. Next, we construct the function $\underline{R}(\underline{x})$:

$$R_i(\underline{x}) = -\propto W(\underline{x}) \frac{\partial v}{\partial x_i} \left\{ 1 + \left(\frac{\partial v}{\partial x_i}\right)^2 \right\}^{-1}$$

$$= -\propto W \sum_{K=1}^{n} a_{ik} x_k \left\{ 1 + \left(\sum_{K=1}^{n} a_{ik} x_k\right)^2 \right\}^{-1}$$

where $\propto = \propto(\underline{x})$ and $-L < \propto < L < \frac{1}{2n}$ . If

$$\left| f_i(\underline{x}) \right| < \left| R_i(\underline{x}) \right|$$

then the trivial solution of the system $\dot{\underline{x}} = \underline{P}\ \underline{x} + \underline{f}\ (\underline{x})$ is asymptotically stable on the whole.  A further note can be made; that is, if the functions $f_i\ (\underline{x})$ are expanded in series, then the last inequality gives an estimate of the coefficients of these series in terms of the arbitrary quantities $b_{ik}$ and in terms of the coefficients, $P_{ik}$ , of the linear system.

Example 6, [3]     Second Order System

We consider the following system, which was originally studied by Zubov:

$$\dot{x}_1 = -x_1 + x_2 + x_1(x_1^2 + x_2^2) ,$$
$$\dot{x}_2 = x_1 - x_2 + x_2(x_1^2 + x_2^2) .$$

(In the next example, (7), we will consider this system with persistent disturbances added to the right-halves of the equations.)

In equation (19), under Zubov's work, we let $\phi = 2\ (x_1^2 + x_2^2)$; that is

$$\frac{\partial v_1}{\partial x_1} (-x_1 + x_2 + x_1 \left[ x_1^2 + x_2^2 \right] ) + \frac{\partial v_1}{\partial x_2} (-x_1 - x_2 + x_2 \left[ x_1^2 + x_2^2 \right] ) =$$

$$= -\ 2(x_1^2 + x_2^2)(1 - V_1) .$$

Assume that

$$V_1 = ax_1^2 + bx_1x_2 + cx_2^2 .$$

Substituting $V_1$ into the partial differential equation gives the following:

$a = c = 1$ and $b = 0$. Thus, $V_1 = x_1^2 + x_2^2$ and
$\dot{V}_1 = -2 (x_1^2 + x_2^2)(1 - x_1^2 - x_2^2)$. Therefore when $V_1 \equiv x_1^2 + x_2^2 < 1$,

$$\dot{V}_1 < 0$$ ; when $V_1 \equiv x_1^2 + x_2^2 \equiv 1$,

$\dot{V}_1 = 0$; and when $V_1 \equiv x_1^2 + x_2^2 > 1$, $\dot{V}_1 > 0$ . Thus, the

boundary of the region of asymptotic stability is $x_1^2 + x_2^2 = 1$ .

### Example 7. [1]     Another Application of "Theorem 3"

We consider the second order case defined by

$$\dot{x} = -x + y + x (x^2 + y^2) + \phi (x,y),$$

$$\dot{y} = -x - y + y (x^2 + y^2) + \psi (x, y).$$

The "first approximation" of the system obtained by discarding the functions
$\phi (x, y)$ and $\psi (x,y)$ has a limit cycle $x^2 + y^2 = 1$, example 6. The
Liapunov function for this "first order" system is chosen to be

$$V = - \ln (1 - x^2 - y^2) ,$$

$$\dot{V} = - 2 (x^2 + y^2) .$$

We let

$$R(x,y) = (x^2 + y^2) \left\{ 1 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 \right\}^{-1/2} ,$$

or

$$R(x,y) = (x^2 + y^2)(1 - x^2 - y^2)(1 + x^2 + y^2)^{-1} .$$

Then, if

$$\left| \phi (x, y) \right| < \left| R (x,y) \right| , \quad \left| \psi (x,y) \right| < \left| R(x,y) \right| ,$$

the original system also has a limit cycle. It should be noted that any

other R(x,y) satisfying the conditions of Theorem 3, in Zubov's work, could

have been used.

**Example 8, [1]**     **Example of Theorem 5**

We consider the curve $S(x, y) = 0$, which is the boundary of a region A: $(0, 0)$

belongs to the closure of A and $S(x, \ddot{y}) < 0$ for $(x, y)$ belonging to A.

If $S(0, 0) \neq 0$     , we then assume that $S(0, 0) = -1$. The general

form of the systems, for which the curve $S(x, y) = 0$ is an integral curve,

was derived by **Erugin**, namely:

$$\dot{x} = f_1(x,y,S) - \frac{\partial S}{\partial y} M(x,y) ,$$

$$\dot{y} = \frac{\partial S}{\partial x} M(x,y) + f_2(x,y,S) ,$$

where M is any continuous function, and $f_1 = f_2 = 0$ when $S = 0$. We separate

from this set of systems the class, for which $(0, 0)$ is asymptotically

stable, and the curve $S(x, y) = 0$ serves as the boundary of the region of

asymptotic stability.

For this purpose, define the following:

$$M(x, y) = - \gamma(x, y) W(x, y)$$

$$f_1(x,y,S) = S(x,y) \left\{ - \gamma(x,y) \frac{\partial W}{\partial y} + S(x,y) \phi(x,y) d_1(x,y) \right\}$$

$$f_2(x,y,S) = S(x,y) \left\{ \gamma(x,y) \frac{\partial W}{\partial x} - S(x,y) \phi(x,y) d_2(x,y) \right\}$$

where

$$\phi(0, 0) = W(0, 0) = \left. \frac{\partial W}{\partial x} \right|_{(0,0)} = \left. \frac{\partial W}{\partial y} \right|_{(0,0)} = 0 .$$

The functions $\phi$ and $W$ are positive definite over the entire plane; the function $\gamma$ is an arbitrary continuously-differentiable function; and the functions $d_1(x, y)$ and $d_2(x, y)$ satisfy the relation

$$d_1(x, y)\left\{ S(x, y)\ \frac{\partial W}{\partial x} - \frac{\partial S}{\partial x}\ W \right\} - d_2(x, y)\left\{ S(x, y)\ \frac{\partial W}{\partial y} - \frac{\partial S}{\partial y}\ W \right\} = 1,$$

but otherwise are arbitrary continuously-differentiable functions. Also, the functions $S$ and $W$ are continuously-differentiable for all values of $x$ and $y$.

We now consider the equation

$$\frac{\partial V}{\partial x}\left\{ S(x,y)\ \left[ S(x, y)\ \phi(x, y)\ d_1(x, y) - \gamma(x, y)\ \frac{\partial W}{\partial y} \right] \right. +$$

$$\left. + \frac{\partial S}{\partial y}\ \gamma(x, y)\ W \right\} + \frac{\partial V}{\partial y}\left\{ - \frac{\partial S}{\partial x}\ \gamma(x, y)\ W(x, y) + S(x, y) \cdot \right.$$

$$\left. \left[ \gamma(x, y)\ \frac{\partial W}{\partial x} - S(x, y)\ \phi(x, y)\ d_2(x, y) \right] \right\} = -\phi(x, y)\left\{ 1 - V \right\}.$$

This equation has a unique continuous solution, defined by the condition

$$1 - V = \exp\left\{ \frac{W(x, y)}{S(x, y)} \right\}, \quad V(0, 0) = 0.$$

Thus by Zubov's theorems, it has been shown that the system derived by Erugin is asymptotically stable and the curve $S(x, y) = 0$ is the boundary of the region of asymptotic stability. Any solution of this system beginning in the region of asymptotic stability is continuable to the semi-axis $t$ belonging to $(-\infty, 0)$ if $\phi$ is bounded.

### Example 9, [1, 2]  Stability of a Periodic Solution

From reference [1], Zubov considered the system.

$$\dot{x} = x + y - x(x^2 + y^2) \equiv f_1(x, y),$$

$$\dot{y} = -x + y - y(x^2 + y^2) \equiv f_2(x, y).$$

We let

$$\phi (x, y) \sqrt{1 + f_1^2 + f_2^2} = 2 (1 - x^2 - y^2)^2 .$$

Thus, the partial differential equation corresponding to the system is given by the following:

$$\frac{\partial V}{\partial x} f_1(x, y) + \frac{\partial V}{\partial y} f_2(x, y) = - (1 - V) \phi \sqrt{1 + f_1^2 + f_2^2}$$

$$= - (1 - V) 2 (1 - x^2 - y^2)^2 .$$

The function V which satisfies this partial differential equation is

$$V = 1 - (x^2 + y^2) \exp \left\{ 1 - x^2 - y^2 \right\} ,$$

where $V = 0$ when $x^2 + y^2 = 1$. Therefore, the circle $x^2 + y^2 = 1$ is a periodic integral curve. From Theorem 7 in the text of this section it follows that the region of asymptotic stability of the periodic solution is the entire plane with the exception of $x^2 + y^2 = 1$ and the origin. The origin is unstable as one may see from the linear approximation

$$\dot{x} = x + y ,$$
$$\dot{y} = - x + y ,$$

whose eigenvalues are $1 \mp i$.

In reference [2], Zubov considers practically the same system, except he extends it to three dimensional phase space. The system is defined by

$$\dot{x} = x + y - x(x^2 + y^2),$$
$$\dot{y} = - x + y - y(x^2 + y^2) ,$$
$$\dot{z} = - z .$$

This system has a periodic motion located in the plane $Z = 0$ and describing a circle of unit radius $x^2 + y^2 = 1$. The partial differential equation for V is given by the following:

$$\frac{\partial V}{\partial x}\left\{x + y - x(x^2 + y^2)\right\} + \frac{\partial V}{\partial y}\left\{-x + y - y(x^2 + y^2)\right\} +$$

$$-\frac{\partial V}{\partial z}z = -\left\{\frac{2(1 + x^2 + y^2)(1 - x^2 - y^2)^2}{x^2 + y^2} + 2\dot{z}^2\right\}(1 - V).$$

This equation differs from the second order case given above, because of the choice of $\phi$. Also, in this case Zubov makes the following substitution:

$$V_1 = -\ln(1 - V),$$

giving the following partial differential equation,

$$\frac{\partial V_1}{\partial x}\left\{x + y - x(x^2 + y^2)\right\} + \frac{\partial V_1}{\partial y}\left\{-x + y - y(x^2 + y^2)\right\} +$$

$$-\frac{\partial V_1}{\partial z}z = -\frac{2\left\{1 - (x^2 + y^2)\right\}^2\left\{1 + x^2 + y^2\right\}}{x^2 + y^2} - 2\dot{z}^2.$$

The solution of this equation is given by

$$V_1 = \frac{\left\{1 - (x^2 + y^2)\right\}^2}{x^2 + y^2} + \dot{z}^2.$$

$V_1$ is defined at all points in the phase space except on $(0, 0, Z)$, that is, the Z-axis; and $V_1$ is zero on $x^2 + y^2 = 1$, $Z = 0$. Consequently, the periodic solution of the third order system is a self-oscillation, the region of attraction of which coincides with the whole phase space, excluding the Z-axis.

### Example 10, [4]    Second Order Example of Equation (35)

The system is defined by

$$\dot{x} = -2x + 2y^4$$

$$\dot{y} = -y$$

Let $\phi$ in equation (35) be given as $\phi = 24(x^2 + y^2)$. The linear partial differential equation for $V_*$ is

$$\frac{\partial V_*}{\partial x} (- 2x + 2y^4) + \frac{\partial V*}{\partial y} (- y) = - 24 (x^2 + y^2) .$$

The solution of this equation can be obtained by the standard techniques of solving Lagrange linear partial differential equations. Thus, we have that

$$V_* = 6x^2 + 12y^2 + 4xy^4 + y^8 \qquad . \quad V_* \longrightarrow \infty \quad \text{as}$$

$\| \underline{x} \| \longrightarrow \infty$ ; therefore, the above system is globally asymptotically stable.

## Example 11, [7, 8]    Rough Systems of Chetaev

This example is concerned with systems which N.G. Chetaev called "Rough Systems". A rough system is a nonlinear system for which the problem of stability can be solved correctly by fairly simple approximate methods. The most interesting of such systems are those for which the problem of the stability of motion reduces to the consideration of linear equations with constant coefficients; these will be considered here. The reason for including this example in this section is because the V-function for the linear approximation is obtained from a partial differential equation of the type found in the text of this section, that is, equation (35).

Consider the following system of equations:

$$\underline{\dot{x}} = (\underline{c} + \epsilon \underline{F}) \underline{x}$$

where $\underline{C}$ is a constant matrix and the elements of $\underline{F}$ are bounded real functions of t, $\underline{x}$ for $X_1^2 + \ldots + X_n^2 \leq A$ and $t \geq t_0$.

The auxiliary system of equations is

$$\underline{\dot{x}} = \underline{c} \, \underline{x}$$

where the eigenvalues of $\underline{C}$, $\lambda_K$, satisfy the following condition: for arbitrary non-negative integers $M_K$, $M_1 \lambda_1 + \ldots + M_n \lambda_n \neq 0$ when $M_1 + \ldots + M_n = 2$. Based on this assumption and a known theorem of

Liapunov, the partial differential equation

$$\sum_{S=1}^{n} \frac{\partial V}{\partial x_s} (C_{s1}x_1 + \ldots + C_{sn} x_n) = - (x_1^2 + \ldots + x_n^2) \equiv U(\underline{x})$$

determines uniquely the following symmetric quadratic form with constant coefficients, $a_{sn}$,

$$V = 1/2 \sum_{S=1}^{n} \sum_{r=1}^{n} a_{sr} x_s x_r .$$

For sufficiently small values of the parameters $\in > 0$, and $A > 0$ and for a small $\mu > 0$, the time derivative $\dot{V}$ by virtue of the equation

$$\dot{\underline{x}} = (\underline{c} + \in \underline{F}) \underline{x}, \text{ satisfies}$$

$$- \dot{V} - \mu (x_1^2 + \ldots + x_n^2) \equiv - \underline{x}_T \left\{ \underline{c}_T \underline{A} + \in \underline{F}_T \underline{A} + \underline{A} \underline{c} + \in \underline{A} \underline{F} + \right.$$

$$\left. + \mu \underline{I} \right\} \underline{x} = \underline{x}_T \underline{H} \underline{x} > 0, \text{ for } \underline{x}_T \underline{x} > 0.$$

Therefore under certain conditions, the asymptotic stability or instability of the undisturbed motion ( $\underline{x} = \underline{0}$) of system $\dot{\underline{x}} = \underline{c} \underline{x}$ corresponds exactly with the asymptotic stability and instability of the corresponding nonlinear system. The quantities $\in$ and A, for which there unconditionally exists such a correspondence, are determined from the n inequalities

$$\begin{vmatrix} h_{11} & \cdots & h_{1p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ h_{p1} & \cdots & h_{pp} \end{vmatrix} \quad > \quad 0, (p = 1, 2, \ldots, n)$$

of Sylvester's theorem, where $\underline{H} = \left[ h_{ij} \right]$ and $\mu$ is sufficiently small.

The bounds for A and $\in$ determined by the above inequalities can be made more precise if in the above partial differential equation one considers in

place of $-(x_1^2 + \ldots + x_n^2)$ some other negative-definite form $U(\underline{x})$ with real coefficients. This fact was also pointed out in the text of this section.

One can give an estimate of the bounds of the largest and smallest deviations of the disturbed variables. For this purpose we consider the extremal values of the derivative $\dot{V}$ on the surface $V = C$. From the above equations, $\dot{V}$ can be written as

$$\dot{V} = \underline{VV}_T \dot{\underline{x}} = \underline{VV}_T (\underline{C}\ \underline{x} + \epsilon\ \underline{F}\ \underline{x})$$

$$= (\underline{VV}_T\ \underline{C}\ \underline{x}) + \epsilon\ \underline{VV}_T\ \underline{F}\ \underline{x}$$

$$= -\underline{x}_T \underline{x} + \epsilon\ \underline{VV}_T\ \underline{F}\ \underline{x} \equiv 1/2\ \underline{x}_T\ \underline{B}\ \underline{x}\ .$$

A result of this work is the following estimate:

$$\underline{x}_T\ \underline{x} \leq c\ \frac{K_n}{K_1}\ \exp\left\{ (\lambda' + \epsilon')\ t \right\}.$$

This inequality gives the square of the radius of the sphere into which at the instant t the point in the perturbed motion $\underline{x}(t)$ will enter if its initial value was $(\underline{x}_0)_T(\underline{x}_0) = C$ for $t_o = 0$, remembering $t > t_o$. The quantities $K_1$ and $K_n$ denote the largest and smallest eigenvalues of $\underline{A}$ in $V = 1/2\ \underline{x}_T\ \underline{A}\ \underline{x}$. Also $\epsilon'$ is a sufficiently small positive constant and $\lambda'$ is the largest root of the equation

$$\left\| \frac{1}{4} \sum_{\ell=1}^{n} \left\{ \frac{\partial\ b_{r\ell}}{\partial x_s} x_\ell + \frac{\partial\ b_{sr}}{\partial x_s} x_r \right\} - \lambda\ a_{sr} \right\| = 0.$$

<u>Example 12, [9]</u>    <u>Homogeneous Atomic Reactor Equation</u>

The reactor equations describing a homogeneous atomic reactor with constant power extraction can be written in the following form:

$$\dot{Z}_1 = -\frac{\alpha}{\tau} Z_2\ ,$$

$$\dot{Z}_2 = \frac{1}{\epsilon} (e^{Z_1} - 1) \equiv f(Z_1).$$

The scalar function $\gamma(\mathbf{z}) = \mathbf{z}_T \, \underline{A}\,(\mathbf{z}) \, \underline{f}\,(\mathbf{z})$ takes the form:

$$\gamma(\mathbf{z}) = - a_{11}\frac{\alpha}{\tau} z_1 z_2 - a_{21}\frac{\alpha}{\tau} z_2^2 + a_{12}\, z_1 f(z_1) + a_{22}\, z_2\, f(z_1) \ .$$

It is evident that by choosing

$$a_{12} = a_{21} = 0 \quad \text{and} \quad a_{11} = \frac{f(z_1)}{z_1}\frac{\tau}{\alpha}\, a_{22}\ ,$$

$\gamma(\mathbf{z})$ will be identically zero.

Thus, the $\underline{A}\,(\mathbf{z})$ - matrix is given by

$$\underline{A}(\mathbf{z}) = \begin{array}{|c|c|} \hline \dfrac{f(z_1)}{z_1}\dfrac{\tau\, a_{22}}{\alpha} & 0 \\ \hline 0 & a_{22} \\ \hline \end{array}$$

The integrability conditions which must be satisfied by the elements of $\underline{A}$ are

$$a_{ij}(z_i, z_j) + z_i\frac{\partial a_{ij}(z_i, z_j)}{\partial z_i} = a_{ji}(z_i, z_j) + z_j\frac{\partial a_{ji}(z_i, z_j)}{\partial z_j}\ .$$

As we can easily see, $\underline{A}$ satisfies these conditions.

For second order systems, the V-function has the form

$$V = \int (z_1 a_{11} + z_2 a_{12}) \, dz_1 + \xi_1(z_2)$$

$$= \int (z_1 a_{21} + z_2 a_{22}) \, dz_2 + \xi_2(z_1)$$

where $\xi_1$ and $\xi_2$ are arbitrary functions. Therefore, if we let $a_{22} = 2$, we have

$$V = \frac{2\tau}{\alpha} \int_0^{z_1} f(s)\,ds + z_2^2$$

$$= \frac{2\tau}{\epsilon\alpha} \int_0^{z_1} (e^s - 1)\, ds + z_2^2\ ,$$

which is a positive definite scalar function if $\dfrac{\tau}{\alpha \epsilon} > 0$ and $Z_1 \geqslant 0$. We conclude that the trivial solution $Z_1 = Z_2 = 0$ is stable for $\alpha, \epsilon, \tau > 0$.

## Example 13, [9]    Third Order System

Let us consider the third order system of Ingwerson, analyzed here by the Zubov-Szego Method, given by the equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = - (x_1 + b x_2)^3 - C x_3 = f(\underline{x}).$$

The corresponding $\gamma(\underline{x})$ function is

$$\gamma(\underline{x}) = \underline{x}_T \underline{A}(\underline{x}) \underline{f}(\underline{x})$$

$$= (a_{11} x_1 + a_{21} x_2 + a_{31} x_3) x_2 + (a_{12} x_1 + a_{22} x_2 + a_{32} x_3) x_3 +$$

$$+ (a_{13} x_1 + a_{23} x_2 + a_{33} x_3) f(\underline{x}).$$

This scalar function is made definite on the trajectories of the system by requiring the discriminant of the equation $\gamma(\underline{x}) = 0$ with respect to $x_3$ to be identically zero. This yields the following two equations:

$$a_{11} x_1 x_2 + a_{21} x_2^2 - a_{13} x_1 (x_1 + bx_2)^3 - a_{23} x_2 (x_1 + bx_2)^3 = 0,$$

$$a_{31} x_2 + a_{12} x_1 + a_{22} x_2 - a_{33} (x_1 + bx_2)^3 - C(a_{13} x_1 + a_{23} x_2) = 0.$$

If we set $a_{13} = a_{31} = 0$ and $a_{23} = a_{32} = 1$, then the first of these two equations gives

$$a_{11} = a_{11}(x) = x_1^2,$$

$$a_{21} = a_{21}(x_1, x_2) = 3bx_1^2 + 3b^2 x_1 x_2 + b^3 x_2.$$

Applying the integrability conditions we obtain

$$a_{12} + \frac{\partial a_{12}}{\partial x_1} x_1 = a_{21} + \frac{\partial a_{21}}{\partial x_2} x_2,$$

which when integrated yields

$$a_{12} = 3b^3 x_2^2 + 3b^2 x_1 x_2 + b x_1^2.$$

From the second equation, obtained by setting the discriminant of $x_3$ equal to zero, we obtain the following results after substituting for $a_{31}$, $a_{12}$, $a_{13}$ and $a_{23}$:

$$a_{33} = b \text{ and } a_{22} = b^4 x_2^2 + C.$$

Thus, the matrix $\underline{A}(x)$ becomes

$$\underline{A}(\underline{x}) = \begin{array}{|c|c|c|}
\hline
x_1^2 & 3b^3 x_2^2 + 3b^2 x_1 x_2 + b x_1^2 & 0 \\
\hline
3b x_1^2 + 3b^2 x_1 x_2 + b^3 x_2^2 & b^4 x_2^2 + C & 1 \\
\hline
0 & 1 & b \\
\hline
\end{array}$$

Integrating $\underline{x}_T \, \underline{A}(\underline{x})$ in a manner similar to example (12) gives the following for V:

$$V(\underline{x}) = \frac{1}{4} \left[ x_1 + b x_2 \right]^4 + \frac{1}{2} \left[ c x_2^2 + 2 x_2 x_3 + b x_3^2 \right],$$

where

$$\dot{V} = x_3^2 (1 - bc).$$

From V and $\dot{V}$, it follows that the trivial solution $\underline{x} = \underline{0}$ of the above third order system is asymptotically stable for b, c $>$ 0 and bc - 1 $>$ 0.

### Example 14, [10]    Second Order System

We consider the second order case given by

$$\dot{x} = y,$$
$$\dot{y} = -ay - ax^3 - x^2 y.$$

The choice for $v_1$ in the equation

$$\dot{v}_1 = \gamma(\underline{x}) \Big/ \beta(v_1)$$

is $\underline{v1 = ax + y}$ . The time derivative of $v_1$ is

$$\dot{v}_1 = a\dot{x} + \dot{y} = -x^2 v_1$$   , Thus, if we choose

$$\gamma(\underline{x}) = -x^2 v_1^2 = -x^2 (ax + y)^2$$   , then

$\beta(v_1) = v_1$. Since $x \ne 0$ and $y \ne 0$, and $y = -ax \ne 0$

are not solutions of the system, then $\gamma(\underline{x})$ is negative definite on the

trajectories. The integral of $\beta(v_1)$ is bounded and nonnegative if $v_1$

is finite, and the integral is unbounded if $\underline{x}$ is unbounded. Therefore,

by Szegö's Theorem, the equilibrium solution $(0, 0)$ is asymptotically

stable.


## Example 15. [10]   Second Order System

Consider the system

$$\dot{x} = y,$$

$$\dot{y} = ax + ax^2 y - y^3 - y,$$

$$a > 0.$$

In this case, choose $v_1 = ax^2 - y^2$. The time derivative is

$$\dot{v}_1 = 2ax\dot{x} - 2y\dot{y} = 2y^2 (1 - v_1)$$   . Now, if

$$\gamma(\underline{x}) = 2y^2 (1 - v_1)^2$$   , then $\beta(v_1) = 1 - v_1$.

Thus,

$$\alpha(v_1) = \int_0^{v_1} \beta(s)\, ds = v_1 - 1/2 \left(v_1^2\right),$$

where

$$\alpha^*(\underline{x}) = (ax^2 - y^2) - 1/2 (ax^2 - y^2)^2,$$

and $\quad \dot{\alpha} *(\underline{x}) = 2y^2 (1 - v_1)^2.$

Since $\dot{\alpha}^*$ is positive semidefinite and $\overset{*}{\alpha}$ is indefinite, then

$x = y = 0$ is unstable.

## Example 16, [10, 11]     Second Order System

This example shows the advantage of Szegö's change of variable technique. The characteristic system for the second order case is

$$\left\{ f_1(\xi, y) - \frac{\partial \xi}{\partial y} f_2(\xi, y) \right\} \left. 1 \middle/ \frac{\partial \xi}{\partial w} \right. = \Theta(w).$$

The system which we consider is

$$\dot{x} = f_1(x,y) = y^3 - x,$$

$$\dot{y} = f_2(x,y) = x - 1/2 \, y.$$

Substituting $f_1$ and $f_2$ into the characteristic system gives

$$\left\{ y^3 - \xi - \frac{\partial \xi}{\partial y} (\xi - 1/2 \, y) \right\} \frac{1}{\partial \xi / \partial w} = \Theta(w).$$

We will assume the form of $\xi = \xi(y, w)$     , $\xi(0, 0) = 0$

and substitute into the above equation. Thus, we let

$$\xi = a \sqrt{w + f(y)}$$

where $f(0) = 0$, but $f(y)$ is otherwise undetermined. The characteristic system becomes

$$\frac{2}{a} y^3 \sqrt{w + f} \quad - 2w - 2f - af' \sqrt{w + f} +$$

$$+ 1/2 \, yf' = \Theta(w).$$

Using Szegö's Theorems as a guideline, we choose $\Theta(w) = -2w$.

If we also choose $f(y)$ such that

$$- 2f + (y/2)f' = 0,$$

and

$$2y^3 \Big/ _a \quad - af' = 0,$$

then we have $f(y) = \pm \, y^4$ and $a^2 = 1/2$. This gives us $\xi$,

namely, $\xi = (1/\sqrt{2}) \sqrt{\omega + y^4}$. If we now go back to the usual form,

$\omega = v$ and $\xi = x$, then we have

$$v = 2x^2 - y^4$$

where

$$\dot{v} = 4 \left\{ x \, \dot{x} - y^3 \dot{y} \right\} = - 2v.$$

Thus, we conclude that the solution, $2x^2 - y^4 = 0$, is globally asymptotically

stable.

In reference [11], $2x^2 - y^4 = 0$ is shown to be a singular solution of the

second order system given in the above discussion. This can be verified if we

write the system in the following form:

$$\frac{dx}{dy} = \frac{2y^3 - 2x}{2x - y} \quad ,$$

where if $u = x/y^2$, then

$$\frac{du}{dy} = \frac{2 - 4u^2}{2uy - 1} \quad .$$

We observe that $\dfrac{du}{dy} = 0$ when $u^2 = 1/2$, that is, when $2x^2 - y^4 = 0$.

The equilibrium points of the system are contained in the solution

curve, $2x^2 - y^4 = 0$. In particular, the equilibrium points are

$$\left( x = 0, \ \pm \sqrt{2}/4 \right) \quad \text{and} \quad \left( y = 0, \ \pm \sqrt{2} \Big/ 2 \right) \quad ;$$

and they lie on the branch of the solution curve given by $y \, |y| = \sqrt{2} \ x$.

The linear approximations of the system in the neighborhood of the

equilibrium points, are

$$\dot{x} = - x$$

$$\dot{y} = - 1/2\, y + x \qquad \text{for } (0, 0) \; ,$$

$$\left.\begin{array}{l} \dot{x} = - (x \mp \sqrt{2}/4) + 3/2\,(y \mp \sqrt{2}/2) \\[2mm] \dot{y} = (x \mp \dfrac{\sqrt{2}}{4}) - 1/2\,(y \mp \sqrt{2}/2\,) \end{array}\right\} \text{for} \quad \begin{array}{l} x = \pm\sqrt{2}/4 \\[2mm] y = \pm\sqrt{2}/2 \end{array} \; .$$

Thus, from linear stability theory we can show that the origin is a stable node and the other equilibrium points are saddle points.

A region of asymptotic stability around (0, 0) can be obtained if we consider the Liapunov function $V = 1/2\,(x^2 + y^2)$, whose derivative along the trajectories of the system is:

$$\dot{V} = - 1/2\, x^2 - 1/2\,(x - y)^2 + x\, y^3 \; .$$

In the region $\mathcal{N}_L$ : $V = 1/2(x^2 + y^2) < 0.295$, $V > 0$ and $\dot{V} < 0$; therefore, all solutions beginning in $\mathcal{N}_L$ tend toward the origin as $t \longrightarrow \infty$ .

**Example 17. [11]    Third Order Case**

Consider the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = - x_1 - x_2 - x_3 + \epsilon(1 - x_1^2 - 2x_1x_2)x_2 +$$

$$+ \epsilon\,(1 - x_1^2)\,x_3 \, , \quad \epsilon > 0 \; .$$

Let a scalar function v be defined by

$$v = - x_1 + \epsilon x_2 - \epsilon x_1^2 x_2 - x_3 \; ,$$

where the total time derivative along the trajectories of the system is

$$\dot{v} = x_1 - \epsilon x_2 + \epsilon x_1^2 x_2 + x_3 = - v \; .$$

The conclusion is that the manifold M on which $v \equiv 0$,

$$M: \quad x_3 = -x_1 + \epsilon x_2 - \epsilon x_1^2 x_2$$

is asymptotically stable. If this equation for M is substituted into the system equations, the third equation, $\dot{x}_3 = f_3(\underline{x})$, becomes an identity and the familiar van der Pol equation is obtained:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \epsilon x_2 - \epsilon x_1^2 x_2 \; .$$

Thus, we conclude that the only equilibrium point, $\underline{x} = \underline{0}$, of the system is unstable. But, the system has one asymptotically stable orbit which lies on the surface , M ,and that orbit is defined by the Van der Pol's equation.

## REFERENCES

(1)  Zubov, V. I., <u>Methods of A. M. Lyapunov and Their Applications</u>, AEC-tr-4439 (Physics), Trans. Issued Oct. 1961.

(2)  Zubov, V. I., <u>Mathematical Methods for the Study of Automatic Control Systems</u>, Trans. by Y. Schorr-Kon, Pergamon Press, 1963.

(3)  Geiss, G., <u>The Analysis and Design of Nonlinear Control Systems via Liapunov Direct Method</u>, Grumman Aircraft Eng. Corp., Research Dept. Rep. G.R.D., No. 121, 1964.

(4)  Margolis, S. G. and Vogt, W. G., "Control Engineering Applications of V. I. Zubov's Construction Procedure for Lyapunov Functions", <u>I.E.E.E. Trans. on Auto. Control</u>, Vol. AC-8, Apr. 1963, No. 2, pp. 104-113.

(5)  Rodden, J. J., "Numerical Applications of Lyapunov Stability Theory", <u>I.E.E.E.., Session IV, paper 2, Auto. Control</u>, 1964, pp. 261-268.

(6)  Kerr, C. N., Discussion of "Control Engineering Applications of V. I. Zubov's Construction Procedure for Liapunov Functions" , <u>I.E.E.E. Trans. on Auto. Control</u>, April 1964, p. 196.

(7)  Chetaev, N. G., "On the Stability of Rough Systems", <u>PMM</u>, Vol. 24, No. 1, 1960, pp. 20-22.

(8)  Chetaev, N. G., "A survey of the Scientific Works of N. G. Chetaev", <u>PMM</u>, Vol. 24, No. 1, 1960, pp. 171-197.

(9)  Szego, G. P., "On the Application of Zubov's Method of Constructing Liapunov Functions for Nonlinear Control Systems", <u>ASME, Jour. of Basic Eng.</u>, June 1963, pp. 137-142.

(10) Szego, G. P., "On a New Partial Differential Equation for the Stability Analysis of Time Invariant Control Systems", <u>J. SIAM Control</u>, Ser. A, Vol. 1, No. 1, 1962, pp. 63-75.

(11) Szego, G. P. and Geiss, G. R., "A Remark on 'A New Partial Differential Equation for the Stability Analysis of Time Invariant Control Systems'", <u>J. SIAM CONTROL</u>, Ser. A, Vol. 1, No. 3, 1963, pp. 369-376.

# BOUNDEDNESS

## AND

# DIFFERENTIAL INEQUALITIES

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# BOUNDEDNESS

## AND

## DIFFERENTIAL

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## BOUNDEDNESS AND DIFFERENTIAL INEQUALITIES

### SUMMARY

In this section we discuss the boundedness of solutions of differential equations.
Boundedness properties are used in the formulation of uniqueness and existence theorems
and in stability studies. The work of Bellman and Yoshizawa make up the major part
of the discussion in this section, although examples are taken from the works of many
other authors.

The application of differential inequalities is discussed since this topic is
certainly strongly related to Liapunov theory. Methods for determining boundedness,
other than differential inequalities, are the monotonicity of the coefficients and the
method of successive approximations.

### INTRODUCTION

We will first define many of the terms used by Yoshizawa and give examples to
prove that the definitions are not equivalent or redundant. Next, we will discuss
in detail the work on boundedness due to Yoshizawa. Then, we will discuss the work
on differential inequalities which is primarily due to Bellman. Finally, we give
many examples of "boundedness" problems studied by various authors.

### Definitions of Yoshizawa [9]

We consider the system defined by

$$\dot{\underline{x}} = \underline{F}(t, \underline{x}) \tag{*}$$

where $\underline{x}$ is an n-vector in $E_n$ and $\underline{F}$ is a given vector field, defined and continuous
in the domain:

$$\Delta : \quad 0 \leq t < \infty, \quad \| \underline{x} \| < \infty.$$

A solution of (*) passing through the initial point (to, $\underline{x}_0$) is denoted by

$$\underline{x}^* = \underline{x}(t; \underline{x}_0, t_0). \tag{**}$$

The following terms are used in the work of Yoshizawa.

(a)  A solution $\underline{x}^*$ issuing from $(t_0, \underline{x}_0)$ is said to be <u>bounded</u> if there exists a positive number B such that $\| \underline{x}^* \| < $ B for $t \geq t_0$. This number depends upon both $\underline{x}_0$ and $t_0$, $B \equiv B(\underline{x}_0, t_0)$.

(b)  The solutions issuing from the point, defined by $t = t_0$, $(\underline{x}_0, t_0)$ to the right are said to be <u>equibounded</u>, if for $\| \underline{x}_0 \| < \alpha$, the B in (a) depends only on $\alpha$ and $t_0$, $B \equiv B(\alpha, t_0)$.

(c)  The solutions are said to be <u>uniformly bounded</u> if for each $t_0$, B is determined only by $\alpha$, $B \equiv B(\alpha)$.

(d)  For the solution $\underline{x}^*$ issuing from $(t_0, \underline{x}_0)$ to the right, if there exist positive numbers B and T such that $\| \underline{x}^* \| < $ B for $t \geq t_0 + T$, the solutions of (*) are said to be <u>ultimately bounded</u>.  B is independent of the particular solution while T may depend upon the solution.  (Here $t_0$ is arbitrary).

(e)  The solutions issuing from $t = t_0$ are said to be <u>equi-ultimately bounded</u> if for $\| \underline{x}_0 \| \leq \alpha$, T in (d) is dependent only on $\alpha$ and $t_0$.

(f)  The solutions are said to be <u>uniformly ultimately bounded</u> if for every $t_0$, T is dependent only on $\alpha$.

(g)  The solutions of (*) are said to be <u>totally bounded</u> (or bounded under constantly acting perturbations), if for any $\alpha > 0$, there exist two positive numbers A and B such that if $\| \underline{x}_0 \| < \alpha$, then $\| \underline{x}^* \| < $ B.  This solution $\underline{x}^*$ is now defined as the solution of

$$\dot{\underline{x}} = \underline{F}(t, \underline{x}) + \underline{H}(t, \underline{x})$$

where $\underline{H}$ is a constantly acting perturbation, $t_0$ is arbitrary, and $t \geq t_0$. The $\underline{H}$ - function must satisfy

$$\| \underline{H}(t, \underline{x}) \| < A \quad ,$$

whenever $\alpha < \| \underline{x} \| < $ B.

The relationships and implications between the various types of boundedness will be presented in "theorem-form" in the subsection devoted to the "Work of Yoshizawa". The following examples indicate that the above definitions are _not_ equivalent or redundant.

Examples [9]

(1) Consider the following system in polar coordinates:

$$\dot{r} = \frac{\dot{g}(t,\Theta)}{g(t,\Theta)} r \quad, \quad \dot{\Theta} = 0 \quad,$$

where

$$g(t,\Theta) = \frac{(1+t)\sin^4 \Theta}{\sin^4 \Theta + (1 - t\sin^2 \Theta)^2} + \frac{1}{1 + \sin^4 \Theta} \cdot \frac{1}{1 + t^2} .$$

The general solution of this system is

$$r = g(t,\Theta_0) r_0, \quad \Theta = \Theta_0.$$

If $\Theta_0 = m\pi$, the solution becomes

$$r = \frac{r_0}{1 + t^2} \quad, \quad \Theta = m\pi \quad ;$$

and if $\Theta_0 \neq m\pi$ (man integer), the solution is

$$r = r_0 \left[ \frac{1 + t}{1 + (t - \xi)^2} + \frac{\xi^2}{1 + \xi^2} \cdot \frac{1}{1 + t^2} \right]$$

$$\Theta = \Theta_0,$$

where $\xi = 1/\sin^2 \Theta_0$.

Every solution is _bounded_; but if $\Theta_0$ is very near $m\pi$, the value of $r$ can be arbitrarily large whenever $t = \xi = \frac{1}{\sin^2 \Theta_0}$, the actual value depending on $\Theta_0$. Thus, the solutions are not _equi-bounded_.

Note

For linear systems, boundedness and equi-boundedness are equivalent; but the solutions being equi-bounded does not imply uniformly bounded solutions, as the

following examples demonstrates.

(2) Let

$$g(t) = \sum_{m=1}^{\infty} \frac{1}{1 + m^4 (t - m)^2}$$

and consider the first order linear differential equation given by

$$x = \frac{\dot{g}(t)}{g(t)} \ x \ .$$

The solution of this equation passing through $(t_0, x_0)$ is

$$x = \frac{x_0}{g(t_0)} \ g(t) \ .$$

If $t_0$ is made sufficiently large, then $g(t_0)$ is sufficiently small. Thus, the solutions are <u>not</u> uniformly bounded, even though the solutions are equi-bounded.

The next examples illustrate that equi-ultimate boundedness is not equivalent to uniform ultimate boundedness.

(3) Consider the first order linear differential equation

$$\dot{x} = - \frac{x}{t + 1} \ .$$

The general solution of this equation is given by

$$x = \frac{(t_0 + 1) \ x_0}{(t + 1)} \ ;$$

and hence for a positive number R, the solutions are clearly equi-ultimately bounded $(t_0 > R)$, but <u>not</u> uniformly ultimately bounded.

(4) The solutions of $\ddot{x} = - x$ are clearly uniformly bounded, but they are not ultimately bounded.

The following examples illustrate that boundedness and Liapunov stability are independent concepts.

(5) The solutions of the equation

$$\dot{x} = 1 \ ,$$

given by $x = c + t$, are obviously unbounded but stable in the sense of Liapunov.

That is, a small change in the initial conditions produces only a small change in the value of x for all future events.

(6) The solution of the equation

$$x = - \left[ 1/2 \; x^2 + \sqrt{x^4 + 4\dot{x}^2} \right] \; x$$

are of the form

$$x = c \; \sin (c \; t + d).$$

Thus, the solutions are bounded, but unstable because of the "C" coefficient of t.

## WORK OF YOSHIZAWA

Yoshizawa has written extensively in the area of boundedness of solutions for ordinary differential equations, [1] to [14]. The technique which he follows for constructing existence, uniqueness and boundedness theorems depends on the construction of a V- function which is similar to the functions employed by Liapunov in his work. (For this reason and because J. LaSalle says that Yoshizawa's work holds the "best promise" in the analysis of time-varying systems, we feel that the following discussion has merit in this report.)

The motivation for the reasoning which Yoshizawa uses is presented in references [1] and [2] . A concept which is very important in the development of Yoshizawa's theory is called the total deviation, or the degree of closeness. For the first order differential equation

$$\frac{dy}{dx} = f(x, y) , \tag{1}$$

the total deviation is defined in the following way. Consider a curve y = g(x); then the total deviation of this curve from the solution of (1) is given by

$$\int_{xo}^{x} \left| g'(x) - f[x, g(x)] \right| \; dx .$$

By making use of this concept, Yoshizawa is able to construct existence and uniqueness theorems for systems which may **not** even have continuously differentiable right-sides.

In $[2]$, the following system of differential equations is considered:

$$\dot{\underline{y}} = \underline{F}(x, \underline{y}),  \tag{2}$$

$$\underline{y} = (y_1, \ldots, y_n)$$

$$\underline{F} = (f_1, f_2, \ldots, f_n),$$

where the $f_i$ are defined in a domain G. The domain G is defined by:

$$0 \leq x \leq a ; \quad |y_i| \leq b_i \quad (i = 1, \ldots, n) ;$$

and the $f_i$ satisfy the properties:

(i)  $f_i$ are Lebesgue measurable in x and are continuous in $y_i$ ;

(ii)  $|f| \leq M_i(x)$, where $M_i(x)$ are Lebesgue integnable over $[0, a]$ ;

(iii)  $S_i(x)$ are the solutions passing through the point P $(x_p, y_p) \in G$ such that the $S_i$ are defined in an interval I, $x_p \in I$, $S_i(x_p) = y_{ip}$ and $(x, S_1(x), \ldots, S_n(x)) \in G$ for $x \in I$ ;

(iv)  $S_i(x) = y_{ip} + \int_{x_p}^{x} f_i \left[ x, S_1(x), \ldots, S_n(x) \right] dx.$

We will now consider a uniqueness theorem for the system in (2). Yoshizawa constructs a V - function by using the above concept of total variation. Let $\underline{P} = (x_p, \underline{y}_p)$ and $Q = (x_q, \underline{y}_q)$ both be points in G such that $x_p < x_q$.

We then denote by $N_{pq}$ the family of all functions that are absolutely continuous in $[x_p, x_q]$, and satisfy

$$\underline{y}(x_p) = \underline{y}_p, \underline{y}(x_q) = \underline{y}_q,$$

where $(x, \underline{y}(x)) \in G$ for all $x \in [x_p, x_q]$.

Thus, if $\underline{y}(x)$ $N_{pq}$, then $\dot{\underline{y}}(x)$ is summable in $[x_p, x_q]$. Now define the function $\underline{D(P,Q)}$ of $\underline{P}$ and $Q$ as follows:

$$D(P, Q) = \inf_{\underline{y}(x) \in N_{pq}} \int_{x_p}^{x_q} \| \dot{\underline{y}}(x) - \underline{f}(x, \underline{y}) \| \, dx \,,$$

where if $x_p = x_q$, then $D(P, Q) = \| \underline{y}_p - \underline{y}_q \|$, and if $x_q > x_p$, then $D(P, Q) = D(Q, P)$.

## Theorem 1    [2]

"The points $P = (x_p, \underline{y}_p)$ and $Q = (x_q, \underline{y}_q)$ belonging to G lie on the same trajectory of (2) if and only if $D(P, Q) = 0$."

## Properties of D(P, Q)

(1) Let points P and Q belong to G, $x_p < x_q$, then for $\underline{y}(x) \in N_{pq}$ we have:

$$\int_{x_p}^{x_q} \| \dot{\underline{y}} - \underline{F}(x, \underline{y}) \| \, dx \geq \| \underline{y}_q - \underline{y}_p \| - \int_{x_p}^{x_q} \| \underline{M}(x) \| \, dx \,,$$

where $\| \underline{M}(x) \| = \left[ M_1^2(x) + \ldots + M_n^2(x) \right]^{1/2}$. From this inequality we get

$$D(P, Q) \geq \| \underline{y}_q - \underline{y}_p \| - \int_{x_p}^{x_q} \| \underline{M}(x) \| \, dx. \tag{3}$$

(2) Let points P, Q, R belong to G and let $x_p \leq x_q \leq x_r$. Thus, we have

$$D(P, R) \leq D(P, Q) + D(Q, R) \,, \tag{4}$$

and

$$\left| D(P, Q) - D(P, R) \right| \leq \| \underline{y}_q - \underline{y}_r \| + \left\| \int_{x_q}^{x_r} \underline{M}(x) \, dx \right\|. \tag{5}$$

(3) $D(P, Q) \geq 0$ and continuous in P and Q.

<u>Definition of the V - function</u>

Let $V(x, \underline{y}) = D(\underline{P}, Q)$ where $\underline{P}$ is the fixed point and $Q$ is the point $(x, \underline{y}(x))$. Then in region G, $V(x, \underline{y})$ is a nonnegative continuous function of x and $\underline{y}$. This V-function is now used in the following uniqueness theorem.

<u>Theorem 2</u>   (uniqueness theorem)   $\begin{bmatrix} 2 \end{bmatrix}$

"(H)   If (i) $f_i$ in (2) satisfy $f_i(x, \underline{0}) = 0$ for all i almost everywhere in $\begin{bmatrix} 0, a \end{bmatrix}$ ;

(ii) $f_i(0, \underline{0}) = 0$ ,

(C)   Then (2) has a unique solution if and only if there exists a $V(x, y)$, defined over G, such that

$$V(x, \underline{0}) = 0 \qquad \text{for } x \in \begin{bmatrix} 0, a \end{bmatrix} ,$$

$$V(x, \underline{y}) > 0 \qquad \text{for } \underline{y} \neq \underline{0} ,$$

and

$$\left| V(x, \underline{y_1}) - V(x, \underline{y}_2) \right| \leq L \, \| \, \underline{y}_1 - \underline{y}_2 \| \qquad +$$

$$+ \left| \int_{x_1}^{x_2} N(x) \, dx \right| ,$$

where $L$ is some constant and $N(x)$ is a nonnegative summable function in $\begin{bmatrix} 0, a \end{bmatrix}$ ."

<u>Example 1</u>

The system is defined by

$$\dot{\underline{y}} = \underline{F}(x, \underline{y}) ,$$

where $\underline{F}$ satisfies $\| \underline{F}(x, \underline{y}) \| \leq N(x) \, \| \underline{y} \|$ . To prove uniqueness, it is sufficient to choose V as

$$V(x, \underline{y}) = \exp \left\{ -2 \, \| \underline{y} \|^2 \int_{o}^{x} N(x) \, dx \right\} .$$

this V satisfies all the conditions in theorem 2.

In reference [3] , Yoshizawa discusses the nonincreasing solutions of

$$y'' = f(x, y, y')$$ (6)

and presents sufficient conditions for the solutions of (6) to be nonincreasing

and tending to zero.

Theorem 3    [3]

(H)  If (i) f (x, y, w) is defined and continuous in the domain $o \leq x < \infty$,

$o \leq y < \infty, -\infty < w < oo$,

(ii) for every c > 0, there exists a continuous function $V(x, y, w) = V_c(x,y,w)$

for the domain $\Delta_c$ :  $o \leq x \leq c, o \leq y \leq c, -k \leq w \leq o$, and if $V_c$ has continuous

first partials in the interior of $\Delta_c$,

(iii) V (x, y, w) > 0 for w $\neq$ 0, and V(x, y, 0) = 0,

(iv) in the interior of $\Delta_c$, V satisfies

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} w + \frac{\partial V}{\partial w} f(x, y, w) \leq 0 ,$$ (7)

(v) y = y(x) is a solution of (6) on the interval $0 \leq a \leq x \leq b$

satisfying the initial conditions $y(a) \geq 0$ and y' (a) > 0,

(vi) $\Delta_c$ is the domain $0 \leq x \leq c, 0 \leq y \leq c, k \geq w \geq 0$, and

V (x, y, w) satisfies

(8)

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} w + \frac{\partial V}{\partial w} f(x, y, w) \geq 0$$

and y'(a) > 0,

(C)  Then hypotheses (i) and (v) imply that $y'(x) \geq 0$ for  [a, b] ; and

hypotheses (i) to (iv) and hypothesis (vi) imply that y'(x) > 0 for (a, b].

The second theorem in [3] cites sufficient conditions for the existence of a

solution of (6).  The proof of the theorem is based on the construction of a

V - function.

<u>Theorem 4</u>  [3]

"(H)  If (i) f (x, y, w) is a continuous function in the domain R: $0 \leq x < \infty$,

$0 \leq y < \infty, -\infty < w < \infty$,

(ii) f (x, 0, 0) $\equiv$ 0 for x $\in$ [0, $\infty$) and f (x, y, 0) $\geq$ 0 for x $\in$ [0, $\infty$)

and y $\in$ [0, $\infty$) ,

(iii) for every c > 0, there exists two functions V (x, y, w) = $V_c$ (x, y, w) and

U(x, y, w) = $U_c$ (x, y, w) defined as follows:

(1)  U and V are positive and continuous,

(2)  U and V converge uniformly to zero as w $\longrightarrow$ - $\infty$, in region

$R_c$: $0 \leq x \leq c, 0 \leq y \leq c, w \leq -k$

(3)  in the interior of $R_c$ they have continuous first partials which

satisfy

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} w + \frac{\partial V}{\partial w} f(x, y, w) \geq 0 ,$$  (9)

and

(10)

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} w + \frac{\partial U}{\partial w} f(x, y, w) \geq 0 ,$$

(C)  Then, for every $y_0$ > 0, there exists at least one solution of (6) defined

[0, $\infty$), satisfying the initial condition y(0) = $y_0$ , and satisfying the inequalities

$$y(x) \geq 0 \text{ and } y'(x) \leq 0."$$  (11)

We now state the main result of reference [3] . This result was obtained

by making use of the previous theorems.

<u>Theorem 5</u>  [3]

(H)  If (i)  f(x, y, w) is a continuous function in R,

(ii)  for every pair of constants $c_1$ and $c_2$, o < $c_1$ < $c_2$ , there exists

a positive continuous function V (y, w) in R* : $c_1 \leq y \leq c_2$ , w $\leq -k$,

(iii)  V has continuous first partial derivatives in the interior of R*,

converging uniformly to zero for $c_1 \leq y \leq c_2$ and when $w \longrightarrow -\infty$ ,

(iv)  V satisfies the following in the interior of R* and for $0 < x < \infty$:

$$\frac{\partial V}{\partial y} w + \frac{\partial V}{\partial w} f(x, y, w) \geq 0 , \tag{12}$$

(v)  for every pair of constants $0 < a_1 < a_2$ , there exists a positive continuous function $U(x, y, w)$ in $\bar{R}$: $\mathscr{J} \leq x < \infty$, $a_1 \leq y \leq a_2$ , $-a_2 \leq w \leq 0$,

(vi)  U converges uniformly to zero for $a_1 \leq y \leq a_2$ , $-a_2 \leq w \leq 0$ when $x \longrightarrow \infty$,

(vii)  in the interior of $\bar{R}$, U has continuous first partial derivatives and satisfies

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} w + \frac{\partial U}{\partial w} f(x, y, w) \geq 0 , \tag{13}$$

(C)  Then, for any solution $y = y(x)$ of (6) on $0 \leq x < \infty$ satisfying $y(x) \geq 0$ and $y'(x) \leq 0$, we have $\lim\limits_{x \to \infty} y(x) = 0$.

In reference [5], Yoshizawa employs V - functions to give sufficient conditions for the <u>ultimate boundedness</u> of solutions of a nonlinear differential equation. He also derives sufficient conditions for a solution of the nonlinear system to be periodic.  The systems which he studies is given by

$$\dot{x} = f(t, x, y)$$

$$\dot{y} = g(t, x, y) , \tag{14}$$

where f and g are continuous in the domain,

$$\Delta_1 : 0 \leq t < \infty , -\infty < x < \infty, -\infty < y < \infty.$$

Before we look at the boundedness of the solutions of (14), consider the following lemmas.

Lemma 1

"Let $A_1$ and $B_1$ be positive constants used to define domain U:

$$| x | < A_1 \quad , \quad | y | < B_1 .$$

Suppose there exists a V - function continuous over the domain $\Delta_2$: $0 \leq t < \infty$, $(x, y) \in U^c$, where $U^c$ is the complement of U. Let V satisfy the following conditions:

(1)   $V(x, y) > 0$,

(2)   $V(x, y) \longrightarrow 0$, uniformly, for y and x respectively when x or y becomes infinite,

(3)   $V(x, y)$ satisfies locally the Lipshitz condition with respect to $(x, y)$ in the interior of $\Delta_2$ ,

(4)   and in the interior of 2, we have

$$\underset{h \rightarrow o}{\text{Lim}} \quad \frac{1}{h} \left\{ V(x + hf(t,x,y),y + hg(t,x,y) - V(x,y) \right\} \geq \epsilon > 0$$

where $\epsilon$ is arbitrarily small but a fixed positive number when x and y are bounded.

Then for any solution of (14), $x = x(t)$ and $y = y(t)$, $\alpha$ and $\beta$ being arbitrary positive numbers, if we have $| x(t_0) | \leq \alpha$ , $| y(t_0) | \leq \beta$ at an arbitrary $t = t_0$ , then there exist two positive numbers $L_1$ and $M_1$ , depending only on $\alpha$ and $\beta$, such that $| x(t) | < L_1$ and $| y(t) | < M_1$ for $t \geq t_0$ ".

Lemma 2

"Under the same assumptions as in Lemma 1, let E be the domain $| x | < A_2$ , $| y | < B_2$ for the arbitrary constants satisfying $A_2 > A_1$ and $B_2 > B_1$. Then for any solution(x, y) such that $(x(to) , y (to)) \in E - \bar{U}$ at $t = to$, we have $(x(t), y(t)) \in \bar{U}$ for some $t > to$."

These lemmas then led Yoshizawa to the following theorems.

## Theorem 6    [5]

"Let the hypotheses of Lemma 1 be satisfied. Then all the solutions of (14) are ultimately bounded. (This means that there exist positive constants $A_3$ and $B_3$ such that $|x(t)| < A_3$, $|y(t)| < B_3$ for any solution of (14) as long as $(x(t_0), y(t_0)) \in E_2$ at $t = t_o$, and for some $T_o$, where $t > T_o$.)"

## Theorem 7    [5]

"Suppose that the same conditions as those in Lemma 1 and the conditions for the uniqueness of solutions in Cauchy's Problem are valid. Moreover suppose that

$$f(t + w, x, y) = f(t, x, y)$$

and

$$g(t + w, x, y) = g(t, x, y)$$

for a positive constant $w$. Then, (14) has at least one periodic solution of period $w$."

## Example 5

Consider the system defined by

$$\ddot{x} + F(\dot{x}) + g(x) = p(t)$$

where $g(x)$ is continuous and $g(x) \operatorname{sgn} x \longrightarrow \infty$ as $|x| \longrightarrow \infty$.

It can be shown that if $F(y)$ is continuous and $F(y) \operatorname{sgn} y \longrightarrow \infty$ as $y \longrightarrow \infty$, and $p(t)$ is continuous and bounded, then all solutions satisfy ultimately

$$|x(t)| < A, \quad |y(t)| < B, \quad y = \dot{x},$$

where $A$ and $B$ are independent of the particular solution chosen.

In applying the methods presented in [5] to this example, we let

$$\dot{x} = y$$

$$\dot{y} = -F(y) - g(x) + p(t).$$

The resultant V-function for this system can then be chosen to be as follows, where a and b are sufficiently large positive constants:

$$V(x, y) = \begin{cases} \exp [u] & , (-\infty < x < \infty; \ y \geqslant b) \\ \exp [u - y + b] & , (x \geqslant a; \ |y| \leqslant b) \\ \exp [u + 2b] & , (x \geqslant a, \ y \leqslant -b) \\ \exp [u + \dfrac{2b}{a} (x + a) - 2b] & , (x \geqslant a, |y| \leqslant -b) \\ \exp [u - 2b] & , (x \geqslant -a, \ y \leqslant -b) \\ \exp [u + y - b] & , (x \leqslant -a, |y| \leqslant b) \end{cases}$$

where $u = u(x, y) = - y^2/_2 - \displaystyle\int_o^x g(x) \ dx.$

In reference [6] , Yoshizawa provides sufficient conditions for every solution of

$$\dot{x} = f(t, x, y)$$

$$\dot{y} = g(t, x, y)$$

to converge to a periodic solution as $t \longrightarrow \infty$, provided the solutions are ultimately bounded.

In reference [7] , Yoshizawa discusses the stability of solutions of a system of differential equations using his V-functions as Liapunov functions. The results of this work coincide with results stated explicitly in other sections of this report.

In reference [8] , Yoshizawa discusses the solutions of the second order boundary-value problems. Yoshizawa summarizes the work done on this problem with reference to the existence and uniqueness of solutions.

Also, in [8] , Yoshizawa considers

$$\underline{\dot{x}} = \underline{F} (t, \underline{x}). \tag{15}$$

The next three theorems give conditions which guarantee the ultimate boundedness of the solutions of (15).

## Theorem 8    [8]

"(H) If (i) D* is the domain: $0 \leqslant t, \ \| \underline{x} \| \geqslant R_o$ , where $R_o$ is sufficiently large,

(ii) there exists a continuous $V(t, \underline{x})$ in $D^*$,

(iii) for any positive number $R \geqslant R_0$, there exists a positive constant $G(R)$ such that $V(t, \underline{x}) \geqslant G(R) > 0$ for $|x| = R$,

(iv) $V(t, \underline{x}) \longrightarrow o$, uniformly, for $\|\underline{x}\| \longrightarrow \infty$,

(v) $V(t, \underline{x})$ satisfies the local Lipshitz condition in $\underline{x}$,

(vi) in the interior of $D^*$, $V$ is absolutely uniformly continuous and we have

$$\underset{h \longrightarrow o}{\underline{Lim}} \frac{1}{h} \left\{ V(t + h, \underline{x} + h \underline{F}(t,\underline{x})) - V(t, \underline{x}) \right\}$$

$\geqslant 0$, almost everywhere,

(C) then given an arbitrary positive number $\propto$, we can find a positive number $\beta > \propto$, such that for any solution of (15) satisfying $\|\underline{x}(t_0)\| \leqslant \propto$ at an arbitrary $t_0 \geqslant 0$, we have for $t \geqslant t_0$

$$\|\underline{x}(t)\| < \beta \quad ''.$$

Theorem 9    [8]

"(H) If (i) the conclusion of theorem 8 is true,

(ii) there exists a function $V(t, \underline{x})$ defined in domain $D^*$,

(iii) $V$ is positive and continuous in $D^*$,

(iv) for any $K > R_0 > 0$, we have $\|\underline{x}\| \leq K$, then $V(t, \underline{x}) \longrightarrow 0$ uniformly as $t \longrightarrow \infty$,

(v) $V$ is locally lipshitzian,

(vi) in the interior of $D^*$, $V$ is absolutely, uniformly continuous and we have

$$\underset{h \longrightarrow 0}{\underline{Lim}} \frac{1}{h} \left\{ V(t + h, \underline{x} + h \underline{F}) - V(t, \underline{x}) \right\} \geqslant 0$$

almost everywhere,

(C) then for any solution of (15) for which $x(t_0) = x_0$ and $\|x_0\| < \gamma$, we have, at some value $t$, say $T \geqslant t_0$, $\|\underline{x}(T)\| \leq R_0$, where $t_0$ is arbitrary.

Theorem 10    [8]

   (H)   If  (i) the assumptions of theorems 8 and 9 are valid

(C)   then there exists a positive constant B (independent of $t_0$ and $\underline{x}_0$) such that any

solution of (15) satisfies ultimately

$$\| \underline{x}(t) \| \; < \; \text{B.}''$$

In $[8]$, Yoshizawa has examples of the above theorems and he also gives a further

extension of theorem 10.

In reference $[9]$ , Yoshizawa considers a theorem which gives the necessary and

sufficient conditions for the solutions of (15) to be bounded.


Theorem 11    [9]

   In order that every solution of (15) be bounded it is necessary and sufficient

that there exists a function $V(t, \underline{x})$ satisfying the following conditions in $\Delta$:

(1)   $V(t, \underline{x}) \; > \; 0$ ,

(2)   $V(t, \underline{x}) \longrightarrow \infty$ uniformly in t, when $\| \underline{x} \| \longrightarrow \infty$,

(3)   for any solution of (15), the function $V(t, \underline{x})$ is a non-increasing function of $t$ .

      And for the condition of equi-boundedness, we further require:

(4)   there exist $\mathcal{K} \; > \; 0$ such that

      $V(t_0, x) \quad \mathcal{K}(K)$ provided   $\underline{x} \; \leq \; K$ .

      the next twelve theorems, from $[9]$, summarize the boundedness results for

equation (15).


Theorem 12    [9]

  (H)   If  (i) $\underline{F}$ is periodic in $t$,

            (ii) solutions issuing from t = 0 are equi-bounded,

            (iii) solutions issuing from t $>$ 0 are simply bounded,

  (C)   then, all solutions are uniformly bounded .

Notation

(1) "$V(t, \underline{x})$ defined in $\overset{*}{\Delta}$" means that V is continuous over $o \leqslant t < \infty, \|\underline{x}\| \geqslant R_0$.

(2) "$D_F V$" = $\underset{h \to o}{\text{Lim}} \frac{1}{h} \left\{ V(t + h, \underline{x} + h \underline{F}) - V(t, \underline{x}) \right\}$

(3) $V(t, \underline{x})$ has "property A" when there exists a $\mathcal{K} = \mathcal{K}(K)$ such that

$V(t, \underline{x}) \leq \mathcal{K}(K)$ when $\|\underline{x}\| \leq K$.

(4) $D_F V$ has "property B" when there is a $\lambda$ such that $D_F V \leq -\lambda (K) < 0$, provided

$\|\underline{x}\| \leq K$.

Theorem 13 $\begin{bmatrix} 9 \end{bmatrix}$

(H) If (i) $V(t, \underline{x})$ is defined in $\Delta$,

   (ii) $V \longrightarrow \infty$ as $\|\underline{x}\| \longrightarrow \infty$,

   (iii) V has property A,

   (iv) in the interior of $\Delta$, we have $D_F V \leqslant 0$,

(C) then the solutions of (15) are equi-bounded."

Theorem 14 $\begin{bmatrix} 9 \end{bmatrix}$

"(H) If (i) $V(t, \underline{x})$ defined in $\overset{*}{\Delta}$,

   (ii) $V \longrightarrow \infty$ uniformly as $\|\underline{x}\| \longrightarrow \infty$,

   (iii) V has property A,

   (iv) in the interior of $\overset{*}{\Delta}$,

we have $D_F V \leqslant 0$,

(C) then the solutions of (15) are uniformly bounded."

Theorem 15 $\begin{bmatrix} 9 \end{bmatrix}$

"(H) If (i) V is defined in $\overset{*}{\Delta}$,

   (ii) $V \longrightarrow \infty$ uniformly as $\|\underline{x}\| \longrightarrow \infty$,

   (iii) V has property A,

   (iv) $D_F V$ has property B,

(C)   then the solutions of (15) are uniformly ultimately bounded."

## Corollary

"(H)   If  (i)   solutions of (15) are uniformly bounded,

(ii)   V is defined in $\Delta^*$ and has property B,

(iii)   $D_F$ V has property B,

(C)   then the solutions are uniformly ultimately bounded."

## Theorem 16   [9]

"(H)   If  (i)   V $\longrightarrow \infty$ uniformly as $\| \underline{x} \| \longrightarrow \infty$,

(ii)   $D_F$ V has property B,

(iii)   there exists some R $\geqslant$ $R_0$ and $\propto$ such that V $\leqslant$ $\propto$ for $\| \underline{x} \|$ = R,

(C)   then, the solutions of (15) are equi-ultimately bounded."

## Theorem 17,   [9]

"(H)   If  (i)   solutions of (15) are uniformly bounded,

(ii)   V is defined in $\Delta^*$ and has property A,

(iii)   there exist $V^*$ (t, $\underline{x}$) $>$ 0 in $\Delta^*$,

(iv)   - $V^*$ has property B,

(v)   $\lim\limits_{t \rightarrow \infty} \left[ D_F V + V^* \right] = 0$  uniformly in any domain $R_0 \leqslant \| \underline{x} \| \leqslant K$,

(C)   then, the solutions of (15) are uniformly ultimately bounded."

## Theorem 18   [9]

"(H)   If  (i)   V defined in $\Delta^*$,

(ii)   V $\longrightarrow \infty$ uniformly as $\| \underline{x} \| \longrightarrow \infty$,

(iii)   there exists R and $\propto$ such that V $\leqslant$ $\propto$ when $\| \underline{x} \|$ = R ,

(iv)   $V^* > 0$ and defined in $\Delta^*$,

(v)   - $V^*$ has property B and $\lim\limits_{t \rightarrow \infty} \left[ D_F V + V^* \right] = 0$ uniformly in

any domain R $\leqslant$ $\| \underline{x} \|$ $\leqslant$ K,

(vi) solutions of (15) are bounded,

(C) then they are ultimately bounded."

## Note

We can replace bounded in (vi) by equi-bounded, then the solutions are equi-ultimately bounded.

## Theorem 19  [9]

"(H)  If,  (i) the system in (15) is of first order,

(ii) solutions are ultimately bounded,

(C) then the solutions of (15) are equi-ultimately bounded."

## Theorem 20  [9]

"(H)  If  (i) (15) is linear,

(ii) solutions are ultimately bounded,

(C) then the solutions are equi-ultimately bounded and equi-asymptotically stable."

## Theorem 21  [9]

"(H)  If  (i)  $\underline{F}$ is periodic,

(ii)  solutions are ultimately bounded,

(C)  then the solutions are equi-ultimately bounded and equi-asymptotically stable."

## Theorem 21  9

"(H)  If  (i)  $\underline{F}$ is periodic,

(ii)  unique solutions of (15) exist for the initial value problems,

(iii)  there exists a $\mathcal{K} > 0$ such that if $\|\underline{x}_0\| \leq B$, $\|\underline{x}(t; \underline{x}_0, 0)\| < \mathcal{K}$.

(iv)  solutions are ultimately bounded, for the bound B,

(C)  then the solutions of (15) are uniformly ultimately bounded."

Theorem 22    [9]

"(H)  If  (i)  $\underline{F}$ is bounded when  $\|\underline{x}\|$ is bounded,

   (ii)  there exists $V \geqslant 0$ in $\triangle$,

   (iii)  $V \longrightarrow \infty$ uniformly as $\|\underline{x}\| \longrightarrow \infty$,

   (iv)  $D_F V \leqslant 0$,

   (v)  R is sufficiently large, then $D_{\underline{F}}V$ has property B for $\|\underline{x}\| \equiv R$,

(C)  then the solutions of (15) are ultimately bounded."

Theorem 23,    [9]

"(H)  If  (i)  $\underline{F}$ is periodic in $t$

   (ii)  solutions issuing from $t = 0$ are equi-bounded and the solutions are

ultimately bounded,

   (iii)  $R_0$ is sufficiently large in $\triangle^*$,

(C)  then there exists a positive $V(t, \underline{x})$ defined in $\triangle^*$ which is continuous and its

first partials are continuous.  $V \longrightarrow \infty$ uniformly as $\|\underline{x}\| \longrightarrow \infty$ and V has

property A.  Also,

$$\dot{V} = \frac{\partial V}{\partial t} \quad (\underline{\nabla V})_T \ \underline{F}(t, \underline{x})$$

has property B."

   In reference [10], Yoshizawa discusses the boundedness of solutions under

perturbations.  He considers the unperturbed system to be given by (15) and the

perturbed system to be

$$\dot{\underline{x}} = \underline{F}(t, \underline{x}) + \underline{H}(t, \underline{x}).  \tag{16}$$

the concept of total boundedness arises in the following theorems; and it is

related to other types of boundedness by these theorems.

**Theorem 24** $\begin{bmatrix} 10 \end{bmatrix}$

"(H) If (i) the solutions of $\overset{\bullet}{\underline{x}} = \underline{A}(t) \, \underline{x}$ are totally bounded,

(C) then the solutions are uniformly ultimately bounded. In fact, $\| \underline{x}(t) \| \longrightarrow 0$

as $t \longrightarrow \infty$.

**Theorem 25** $\begin{bmatrix} 10 \end{bmatrix}$

"(H) If (i) $R > 0$, constant, and sufficiently large,

　　　(ii) $\triangle^*$ is defined by,

$\triangle^*:$　$0 \leq t < \infty;$　$\| \underline{x} \| \geqslant$　$R,$

　　　(iii) V is a positive continuous function in $\overset{*}{\triangle}$,

　　　(iv) V has property A and $V \longrightarrow \infty$ uniformly as $\| \underline{x} \| \longrightarrow \infty$,

　　　(v) V is locally lipshitzian in $\underline{x}$,

　　　(vi) $D_F \, \phi$ has property B,

(C) then the solutions of (15) are totally bounded."

**Theorem 26** $\begin{bmatrix} 10 \end{bmatrix}$

"(H) If (i) $\underline{F}$ in (15) belongs to $C'$, with respect to $\underline{x}$,

　　　(ii) $\underline{F}$ is periodic in $t$,

　　　(iii) solutions issuing from $t = 0$ are equibounded and if the solutions

are ultimately bounded,

(C) then they are totally bounded."

**Definition**

For a given positive function $f( \| \underline{x} \| )$, the solutions of (15) are said to be

ultimately bounded under constantly acting perturbations of order $f( \| \underline{x} \| )$ if there

exist two positive constants $\alpha$ and $\beta$ such that

$$\| \underline{H} (t, \underline{x}) \| < \propto f ( \| \underline{x} \| ) \text{ for } \| \underline{x} \| \geq B; \text{ and then we have}$$

$$\lim_{t \longrightarrow \infty} \| \underline{x} (t; \underline{x}_0, t_0) \| < B,$$

where $\underline{x}$ is any solution of (16).

## Theorem 27 [10]

"(H) If (i) V is positive and continuous in $\overset{*}{\Delta}$,

(ii) V has property A and V $\longrightarrow \infty$ uniformly as $\| \underline{x} \| \longrightarrow \infty$,

(iii) $V (t, \underline{x}) - V (t, \underline{x}_1) \leq K \sqrt{\| \underline{x} - \underline{x}_1 \|}$

where $\underline{x}$ and $\underline{x}_1$ are sufficiently close such that K is only a function of $\| \underline{x} \|$,

(iv) $D_F V \leq - G( \| \underline{x} \| )$, where G is a positive continuous function for $\| \underline{x} \| \geq R_3$,

(v) $K^2 f ( \| \underline{x} \| ) = 0 (G^2 ( \| \underline{x} \| ))$ as $\| x \| \longrightarrow \infty$,

(C) then the solutions of (15) are ultimately bounded under constantly acting

perturbations of order $f ( \| \underline{x} \| )$."

In references [11, 12, 13], Yoshizawa discusses the boundedness properties

of $\overset{\bullet}{\underline{x}} = \underline{F}(t, \underline{x})$ in more detail.

In reference [14], Yoshizawa considers the following second order system:

$$\overset{\bullet\bullet}{x} = F (t, x, \overset{\bullet}{x}) . \tag{17}$$

The following theorem gives sufficient conditions for bounded solutions of (17).

## Theorem 28 [14]

"(H) If (i) two functions $\underline{w} (t)$ and $\overline{w} (t)$ are defined on $o \leq t < \infty$ and

belong to $c^2$,

(ii) $\underline{w}$ and $\overline{w}$ are bounded along with their derivatives,

(iii) $\underline{w} \leq \overline{w}$, for all t

$\overset{\bullet\bullet}{\overline{w}} \leq F (t, \overline{w}, \overline{w})$, for all t

$\overset{\bullet\bullet}{\underline{w}} \geq F (t, \underline{w}, \underline{w})$, for all t

(iv) D is the domain where $o \leq t < \infty$ and $\underline{w}(t) \leq x \leq \overline{w}(t)$,

(v) domain $D_1$ is defined by $(t, x) \in D$ and $y \geqslant k$, if domain $D_2$ is defined by $(t, x) \in D$ and $y \leq - k$, where k is a sufficiently large positive number and $y = \overset{\bullet}{x}$,

(vi) there exists two positive continuous functions $V_1(t, x, y)$ in $D_1$ and $V_1(t, x, y)$ in $D_1$,

(vii) $V_1$ , $V_2 \leq a(|y|)$, where "a" is a positive continuous function,

(viii) $V_1, V_2 \longrightarrow \infty$ uniformly as $|y| \longrightarrow \infty$,

(ix) $V_1, V_2$ satisfy the following in the interior of $D_1$ and $D_2$:

$$\overset{\bullet}{V_1} = \lim_{h \longrightarrow o^+} \frac{1}{h} \left\{ V_1(t + H, x + hy, y + h F) - V_1(t, x, y) \right\}$$

$$\geqslant o,$$

$$\overset{\bullet}{V_2} = \overline{\lim_{h \longrightarrow o^+}} \frac{1}{h} \left\{ V(t + h, x + hy, y + h F) - V_2(t, x, y) \right\} \leq o$$

(C) then the equation (17) has a bounded solution". Yoshizawa uses theorem 28 to prove the existence of periodic solutions for a wide class of equations.

## Differential Inequalities

One way to look at the direct method of Liapunov is the following:
"it depends basically on the fact that a function satisfying the inequality

$$\overset{\bullet}{m}(t) \leq w(t, m(t)), m(t_\theta) = r_o,$$

is majorized by the maximal solution of the equation

$$\overset{\bullet}{r} = w(t, r), r(t_o) = r_o."$$

this comparison principle enables one to study various problems of differential equations. Because of this fact we make an excursion into the area of differential inequalities.

The topics which can be studied through the use of differential inequalities are upper and lower bounds for solutions of differential equations, the unboundedness of

solutions, asymptotic behavior of solutions, existence of oscillatory solutions, and many other topics.

The following inequality, known as Gronwall's inequality, is one of the simpler and more useful ones.

## Theorem 1

"Let $0 \leq g(t)$ on $[a, b]$ and let c and m be positive constants, then if

$$g(t) \leq c + m \int_a^t g(s) \; ds \text{ on } [a, b], \tag{1}$$

then

$$g(t) \leq c \; \exp[mt] ." \tag{2}$$

this result can be easily generalized to the following case:

"if u(t) and v(t) are non-negative continuous functions on $[a, b]$, $c > 0$, and

$$v(t) \leq c + \int_a^t v(s) \; u(s) \; ds, \quad [a, b],$$

then $v(t) \leq c \; \exp\left[\int_a^t u(s) \; ds\right]$ ; and if $c = 0$, then $v(t) = 0$ for $[a, b]$."

In reference 23 , Conlan's generalization of the above theorem is for the case where v(x) is a vector-valued function, $v(\underline{x}) = v(x_1, \ldots, x_n)$.

As an example of the application of Gronwall's inequality, we consider a theorem from Bellman's book, $[15]$.

## Theorem 2    $[15]$

"All the solutions of $\ddot{u} + (1 + x(t))u = 0$ are bounded provided that

$$\int^\infty | \dot{x}(t) | \; dt < \infty,$$

$$x \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty."$$

## Proof

Multiply $\ddot{u} + (1 + x(t)) u = 0$ by $\dot{u}$ and integrate by parts to obtain

$$\frac{(\dot{u})^2}{2} + \frac{u^2}{2} (1 + x(t)) - \frac{1}{2} \int_0^t \dot{x}(t_1) u^2 \, dt_1 = c_2.$$

We can now take to sufficiently large so that

$$1 + x(t) \geq 1/2 \quad \text{for} \quad t \geq t_0 ;$$

thus, we have

$$u^2 \leq 4 |c_2| + 2 \int_0^t |\dot{x}(t_1)| u^2 \, dt_1$$

$$c_3 + 2 \int_0^t |\dot{x}(t_1)| u^2 \, dt_1.$$

By Gronwall's inequality, we have

$$u^2 < c_3 \exp\left[ 2 \int_{t_0}^{\infty} |\dot{x}(t_1)| \, dt_1 \right].$$

Another example of this technique is also a theorem which comes from [15].

## Theorem 3    [15]

"If $\ddot{u} + a(t) u = 0$ and

$$\int^{\infty} t |a(t)| \, dt < \infty, \text{ then } \lim_{t \to \infty} \dot{u} \text{ exists.}"$$

## Proof

Integrating $\ddot{u} + a(t) u = 0$ twice between the limits $1$ and $t$ gives

$$u = c_1 + c_2 t - \int_1^t (t - t_1) a(t_1) u(t_1) \, dt_1.$$

For t $>$ 1, we have

$$\frac{|u|}{t} \leq |c_1| + |c_2| + \int_1^t t_1 |a(t_1)| \frac{|u(t_1)|}{t_1} dt;$$

By Gronwall's inequality we obtain

$$\frac{|u|}{t} \leq (|c_1| + |c_2|) \exp\left[\int_0^t t_1 |a(t_1)| dt_1\right] \leq c_3.$$

Since

$$|\dot{u}| = \left| c_2 - \int_1^t a(t_1) u(t_1) dt_1 \right|,$$

then from $|u| \leq t c_3$ we have

$$|\dot{u}| \leq |c_2| + c_3 \int_1^t t_1 |a(t_1)| dt_1.$$

Thus since $\int_1^t t_1 |a(t_1)| dt$ converges, $\lim_{t \to \infty} \dot{u}$ exists.

Another linear, time-varying, second order system for which the application of Gronwall's inequality is very important is given in theorem 4. (We state the theorem but do not give the proof.)

<u>Theorem 4</u>  $[15]$

"If all the solutions of $\ddot{u} + a(t) u = 0$ satisfy $\int_a^b u^2 dt < \infty$, then all the solutions of $\ddot{u} + [a(t) + b(t)] u = 0$ also satisfy $\int_a^b u^2 dt < \infty$ provided $|b(t)| < c, t \geq 0$."

Gronwall's inequality and its generalization, can be applied to <u>nth</u> order systems

as well as second order systems.  In some cases, it may be more convenient to use a different norm than the Euclideon norm.  Even though this may be the case in what follows, we will <u>not</u> discuss the differences between norms; but will assume that all the norms are equivalent for the operations performed.

<u>Theorem 5</u>    [15]

"If all solutions of $\dot{y}$ = $\underline{A}$ $\underline{y}$, where $\underline{A}$ is a constant matrix, are bounded as t $\longrightarrow$ $\infty$, then the same is true of the solutions of

$$\dot{\underline{z}} = \left[ \underline{A} + \underline{B}(t) \right] \underline{z}$$

provided that

$$\int^{\infty} \| \underline{B} \| \ dt < \infty."$$

<u>Proof</u>

Rewrite the equation $\dot{\underline{z}}$ = $\underline{A}$ $z$ + $\underline{B}(t)$ $\underline{z}$

as

$$\underline{z} = \underline{y} + \int_{o}^{t} \underline{Y} (t - t_1) \underline{B}(t_1) \underline{z}(t_1) \ dt_1$$

where $\underline{y}$ is the solution of $\underline{y}$ = $\underline{A}$ $\underline{y}$ satisfying $\underline{y}(0)$ = $\underline{z}(0)$, and $\underline{Y}$ is the matrix solution of $\dot{\underline{Y}}$ = $\underline{A}$ $\underline{Y}$, $\underline{Y}(0)$ = $\underline{I}$.

Now, let C = max ( $\sup\limits_{t \geqslant o} \| \underline{y} \|$ , $\sup\limits_{t \geqslant o} \| \underline{Y} \|$ ) then

$$\| \underline{z} \| \leqslant \| \underline{y} \| + \int_{o}^{t} \| \underline{Y} (t - t_1) \| \cdot \| \underline{B}(t_1) \| \cdot \| \underline{z}(t_1) \| \ dt_1$$

$$\leq c_1 + c_1 \int_{o}^{t} \| \underline{B}(t_1) \| \cdot \| \underline{z}(t_1) \| \ dt_1.$$

By Gronwall's inequality we have

$$\| \underline{z} \| \leq c_1 \exp \left[ c_1 \int_0^t \| \underline{B} \| \, dt_1 \right]$$

$$\leq c_1 \exp \left[ c_1 \int_0^\infty \| \underline{B} \| \, dt_1 \right] .$$

Thus, since $\int_0^\infty \| \underline{B} \| \, dt$ is bounded, $\| \underline{z} \|$ is bounded.

## Theorem 6    [15]

"If all the solutions of $\overset{\bullet}{\underline{y}} = \underline{A} \, \underline{y}$ approach zero as $t \longrightarrow \infty$, then all solutions of $\overset{\bullet}{\underline{z}} = (\underline{A} + \underline{B}(t)) \, \underline{z}$ approach zero as $t \longrightarrow \infty$ provided that $\| \underline{B}(t) \| \leq c_1$ for $t > t_0$, where $c_1$ is a constant depending on $\underline{A}$."

The <u>proof</u> of this theorem also is dependent upon Gronwall's inequality.

In reference [15], Bellman gives several other theorems dealing with the system

$$\overset{\bullet}{\underline{z}} = (\underline{A} + \underline{B}(t)) \, \underline{z}.$$

The proofs of these theorems depends upon the application of Gronwall's inequality.

## OTHER EXAMPLES OF THE APPLICATION OF DIFFERENTIAL INEQUALITIES

The following discussion will be a brief outline of the contributions of many authors to the fields of differential inequalities and differential equations.

In reference [32], Viswanatham generalizes Gronwall's inequality. We present this work in theorem-form.

## Theorem 7    [32]

"If $\phi(x) \leq \eta + \int_{x_0}^x f(s, \phi(s)) \, ds$

where f (x, y) is continuous and monotonic increasing in y in the region

R defined by $|x - x_0| \leq a$, $|y - n| \leq b$, where a and b are positive real numbers,

and $\phi(x)$ is continuous in $|x - x_0| \leq a$, then $\phi(x) \leq \lambda(x)$, where $\lambda(x)$ is the

maximal solution of $\dot{z} = f(x,z)$ through the point $(x_0, n)$ for $x \geq x_0$."

Theorem 8    [32]

"Assume that the conditions on $f(x,z)$ in theorem 7 are valid and assume that

$$\phi(x) \geq \eta + \int_{x_0}^{x} f(s, \phi(s))\, ds ,$$

then $\phi(x) \geq \psi(x)$, where $\psi(x)$ is the minimal solution of the differential equation

$\dot{z} = f(x,z)$ through $(x_0, n)$ for $x_0 \leq x \leq x_0 + a$."

Corollary 1

"Under the conditions on f(x, y) in theorem 7,

if     $$\overline{\Phi}(x) \leq \psi(x) + \int_{x_0}^{x} f(s, \overline{\Phi}(s))\, ds$$

then $\overline{\Phi}(x) \leq \psi(x) + \lambda(x)$ for $x \geq x_0$, where $\lambda(x)$ is the maximal solution of

$\dot{z} = f(x, z + \psi(x))$ through $(x_0, 0)$ and as far as the maximal solution exists."

Corollary 2 gives a similar result corresponding to theorem 8.

Gronwall's inequality follows from theorem 7 if $f(x, y) = |f(x)| \, y$, $x_0 = 0$.

If $f(x, y) = v(x) g(y)$, where $v(x)$ is non-negative and $g(y)$ is monotonic increasing

in y, then we get the inequality mentioned earlier in the text.

The theorems 7 and 8 are useful in providing estimates on the closeness of approxi-

mate solutions to the actual solution, and providing bounds or the norms of solutions

of differential equations. Applications of these theorems are provided by Bihari [16] ,

Langenhop [28] , and Coddington and Levinson [22] .

In reference [21], Choy-Tak Taam derives some criteria for the boundedness of the solutions of certain nonlinear differential equations using Gronwall's inequality. The system he considers is given by

$$\frac{d\,(r(x)y')}{d\,x} + \sum_{i=1}^{n} p_i(x)\, y^{2i-1} = 0.$$

In reference [21], Choy-Tak Taam derives sufficient conditions for the solutions of the equation

$$\frac{d\,(r(x)\, y')}{d\,x} + q(x)\, y = f(x,\, y)$$

to be bounded, where $f(x,\, y)$ is a "small" nonlinear term. He assumes Lebesgue functions and that $f(x,\, y)$ is lipshitzian in y.

In reference [25], Kolodner derives expressions for the upper and lower bounds for the upper and lower bounds for the solutions of the Ricatti equation:

$$\dot{x} + x^2 = f(t).$$

Kolodner makes use of differential inequalities in his proofs.

In reference [31], Utz states and proves five theorems concerning the boundedness of nonlinear, second order differential equations of the type $\ddot{x} + f(x,\, \dot{x}) + g(x) = 0.$ His theorems are generalizations of the following statement:

"If d > 0 and e > 0 are real constants, then a solution $x(t) \not\equiv 0$ of the linear equation $\ddot{x} + d\dot{x} + ex = 0$ is oscillatory or monotonically approaches zero."

In reference [24], Hochstadt extends the following theorem of Liapunov.

## Theorem 9    [24]

"Let $p(t) \not\equiv 0$ be nonnegative, piecewise continuous, periodic function of period T. Then all solutions of

$$\ddot{y} + p(t)\, y = 0$$

are bounded for all $t$ if

$$T \int_0^T p(t) \, dt \le 4."$$

Hochstadt's theorem is:

## Theorem 10    [24]

"Let $p(t) \not\equiv 0$ be nonnegative, integrable, periodic function of period $T$ and $\bar{\lambda}$ be the smallest eigenvalue of the boundary - value problem

$$\ddot{y} + \lambda \, p(t) \, y = 0$$

$$y(0) + y(T) = 0$$

$$\dot{y}(0) + \dot{y}(T) = 0.$$

If $T \int_0^T p(t) \, dt \le 4$, then $\bar{\lambda} > 1."$

In reference [33], Waltman establishes a criterion for the oscillation of all solutions of

$$\ddot{y} + a(t) \, f(y) = 0.$$

He gives three theorems dealing with this problem, where the following conditions are placed on $f(y)$:

(1) $f(0) = 0$; $f(y) \ne 0$, $y \ne 0$

(2) $\dfrac{df}{dy}$ continuous and nonnegative.

In reference [30], Trench considers the behavior of the solutions of the differential equation

$$\ddot{x} + \left[ f(t) + g(t) \right] u = 0$$

as $t \longrightarrow \infty$. He assumed that the solutions of $\ddot{z} = f(t) \, z$ are known.

In reference [26], Lakshiborth, applies Gronwall's inequality to derive criteria for the boundedness and unboundedness of solutions of

$$\frac{dy}{dx} = f(x, y).$$

In some of his proofs, he uses Liapunov-like functions.

Finally, in reference [27], Lakshmikantham provides bounds on the norms of the solutions of

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}), \ \underline{x}(t_o) = \underline{x}_o, \ t_o \geqslant 0.$$

He uses a "test function" in his theorems and proofs which is strongly related to a Liapunov function. Also, he studies the system

$$\dot{\underline{x}} = \underline{A}(t) \underline{x} + \underline{F}(t, \underline{x}) \quad \underline{f}(t, \underline{x})$$

with regards to stability and asymptotic stability.

# REFERENCES

1. Yoshizawa, T. and Hayashi, K., "On the Uniqueness of Solutions of a System of Ordinary Differential Equations", Mem. Coll. Sci. Kyoto, A., 26, pp. 19-29, 1950.

2. Yoshizawa, T. and Hayashi, K., "New Treatise of Solutions of a System of Ordinary Differential Equations and Its Application to the Uniqueness Theorem", Mem. Coll. Sci. Kyoto, A., 26, pp. 225-233, 1951.

3. Yoshizawa, T., "Note on the Non-increasing Solutions of y = f (x,y,y)", Mem. Coll. Sci. Kyoto, A., 27, pp. 153-162, 1952.

4. Yoshizawa, T., "On the Evaluation of the Derivatives of Solutions of y = f (x,y,y)", Mem. Coll. Sci. Kyoto, A., 28, pp. 27-32, 1953.

5. Yoshizawa, T., "On the Non-linear Differential Equation", Mem. Coll. Sci. Kyoto, A., 28, pp. 133-141, 1953.

6. Yoshizawa, T., "On the Convergence of Solutions of the Non-linear Differential Equation", Mem. Coll. Sci. Kyoto, A., 28, pp. 143-151, 1953.

7. Yoshizawa, T., "On the Stability of a System of Differential Equations", Mem. Coll. Sci. Kyoto, A., 29, pp. 27-33, 1955.

8. Yoshizawa, T., "Note on the Solutions of a System of Differential Equations", Mem. Coll. Sci. Kyoto, A., 29, pp. 249-273, 1955.

9. Yoshizawa, T., "Note on the Boundedness and the Ultimate Boundedness of Solutions of $\underline{X} = \underline{F}$ $(t,\underline{x})$", Mem. Coll. Sci. Kyoto, A., 29, pp. 275-291, 1955.

10. Yoshizawa, T., "Appendix to the Paper 'Note on the Boundedness and the Ultimate Boundedness'", Mem. Coll. Sci. Kyoto, A., 30, pp. 217-226, 1957.

11. Yoshizawa, T., "On the Necessary and Sufficient Conditions for the Uniform Boundedness of Solutions of $\underline{X} = \underline{F}$ $(t,\underline{x})$", Mem. Coll. Sci. Kyoto, A., 30, pp. 217-226, 1957.

12. Yoshizawa, T., "Note on the Equi-ultimate Boundedness of Solutions of $\underline{X} = \underline{F}$ $(t,\underline{x})$", Mem. Coll. Sci. Kyoto A., 31, pp. 211-217, 1958.

13. Yoshizawa, T., "On the Equiosymptotic Stability in the Large", Mem. Coll. Sci. Kyoto, A., 32, pp. 171-180, 1959.

14. Yoshizawa, T., "Existence of a Bounded Solution and Existence of a Periodic Solution of the Differential Equation of the Second Order", Mem. Coll. Sci. Kyoto, A., 33, pp. 301-308, 1960.

15. Bellman, R., Stability Theory of Differential Equations, McGraw-Hill Book Company, Inc., New York, 1954, 166 pp.

16. Bihari, I., "A Generalization of a Lemma of Bellman and its Application to Uniqueness Problem of Differential Equations", _Acta. Math Acad. Sci. Hungary_, 7, pp. 81-94, 1956.

17. Brauer, F., "Bounds for Solutions of Ordinary Differential Equations", _Proc. Amer. Math. Soc._, 14, pp. 36-43, 1963.

18. Brauer, G., "Remarks on a Paper by Utz," _Proc. Amer. Math. Soc._, 9, pp. 34-36, 1958.

19. Bridgland, T. F., Jr., "Asymptotic Equilibrica in Homogeneous and Non-homogeneous Systems", _Proc. Amer. Math. Soc._, 15, pp. 131-135, 1964.

20. Taam, Choy-Tak, "Criteria of Boundedness of the Solutions of Nonlinear Differential Equations, _Proc. Amer. Math. Soc._, 6, pp. 377-386, 1955.

21. Taam, Choy-Tak, "The Boundedness of the Solutions of a Nonlinear Differential Equation", _Proc. Amer. Math. Soc._, 5, pp. 122-125, 1954.

22. Coddington, E. A. and Levinson, N., _Theory of Ordinary Differential Equations_, McGraw-Hill Book Company, Inc., 1955, 429 pp.

23. Conlan, J., "A Generalization of Gronwalls' Inequality", _Contr. Diff. Eq._, Vol 3, pp. 37-42, 1964.

24. Hochstadt, H., "A New Proof of a Stability Estimate of Lyapunov," _Proc. Amer. Math. Soc._, 14, pp. 525-526, 1963.

25. Kolodner, I. I., "Bounds for Solutions of the Riccati Equation", _Amer. Math Mo._, 68, pp. 766-769, 1961.

26. Lakshihanth, V., "On the Boundedness of Solutions of Nonlinear Differential Equations", _Proc. Amer. Math. Soc._, 8, pp. 1044-1048, 1957.

27. Lakshmikantham, V., "Upper and Lower Bounds of the Norm of Solutions of Differential Systems", _Proc. Amer. Math. Soc._, 14, pp. 509-513, 1963.

28. Langenhop, C. E., "Bounds on the Norm of a Solution of a General Differential Equations," _Proc. Amer. Math. Soc._, 11, pp. 795-799, 1960.

29. Sell, G. R., "Boundedness of Solutions of Ordinary Differential Equations and Lyapunov Functions", _Joun. M.A.A._, 9, pp. 477-490, 1964.

30. Trench, W. F., "On the Asymptotic Behavior of Solutions of Second Order Linear Differential Equations", _Proc. Amer. Math. Soc._, 14, pp. 12-14, 1963.

31. Utz, W. R., "Properties of Solutions of Certain Second Order Nonlinear Differential Equations", _Proc. Amer. Math. Soc._, 8, pp. 1024-1028, 1957.

32. Viswanatham, B., "A Generalization of Bellman's Lemma", _Proc. Amer. Math. Soc._, 14, pp. 15-18, 1963.

33. Waltman, P., "Oscillations of Solutions of a Nonlinear Equation," _Siam Review_ 5, pp. 128-131, 1963.

SECTION TEN

STABILITY

OF

NONAUTONOMOUS SYSTEMS

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## STABILITY OF NONAUTONOMOUS SYSTEMS

### SUMMARY

In this section we consider some of the problems which occur in the stability analysis of nonautonomous systems. Most of the work which is discussed here deals with linear systems, although there is some discussion concerning nonlinear systems.

This section is divided into four parts; theorems on continuability and boundedness, theorems on the stability of linear, nonautonomous systems, theorems on the stability of nonlinear, nonautonomous systems, and methods of constructing Liapunov functions. In this section there is no compendium of examples, as such; the examples are given throughout the text to aid in the discussion of the material. However, it should be <u>noted</u> that there are a few other nonautonomous systems scattered throughout the rest of the report. Also, because of the "Method of Construction" of this report, there are examples of time - varying systems in the miscellaneous section.

## (I) THEOREMS ON CONTINUABILITY AND BOUNDEDNESS

Let us consider the system of differential equations:

$$\dot{\underline{x}} = - \underline{f}\,(\underline{x},\,t)\,, \tag{1}$$

where

$$\underline{x}\,(0) = \underline{x}o, \quad 0 \leqslant t < \infty\,, \tag{2}$$

and $\underline{f}$ is continuous for all $o \leqslant t < \infty$ and $\|\underline{x}\| < \infty$.

We shall use two norms, denoted as follows:

$$|\underline{x}| = |x_1| + |x_2| + \ldots + |x_n|\,,$$

and

$$\|\underline{x}\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}.$$

Let $V\,(\underline{x},\,t)$ be a scalar function defined for $0 \leqslant t < \infty$ and for all $\underline{x}$ with the following properties:

   (i)  $V$ is continuous in $\underline{x}$ and $t$;

   (ii) $V$ has one - sided derivatives with respect to $\underline{x}$ and $t$;

(iii) $\lim\limits_{|x| \to \infty} V(\underline{x}, t) = \infty$ for all t; and $\|\underline{x}\| < \infty$;

(iv)  V is positive definite.

Let $\dot{V}$, defined as

$$\dot{V} = \frac{\partial V}{\partial t} + (\underline{\nabla V})_T \underline{f}(\underline{x}, t),$$

be bounded (whenever it is necessary) as follows:

$$\dot{V}(\underline{x}, t) \leq \omega(t, V(\underline{x}, t)), \tag{3}$$

$$\dot{V}(\underline{x}, t) \geq \Theta(t, V(\underline{x}, t)), \tag{4}$$

where $\Theta$ and $\omega$ are continuous for

$o \leq t < \infty$, $|\underline{x}| < \infty$, and $\omega(t, o) = \Theta(t, o) = o$.

The following theorems, 1 and 2, are concerned with the continuity of the solutions of the initial - value problem in (1) and (2) as t becomes large.

Lemma 1 (By Conti)  [1]

"Let $\underline{x}(t)$ be a solution of (1).  Define m(t) by the following:

m(t) = V (t, $\underline{x}$ (t)) , ro = m(o) = V(o, $\underline{x}_o$).

Let (3) be satisfied by V, and let r(t) be the maximum solution of the equation

$$\dot{r} = \omega(t, r), \quad r(0) = r_o. \tag{5}$$

Then $\underline{x}(t)$ can be continued to the right(as a function of t) as far as r(t) exists, and for all t for which m(t) $\leq$ r(t)".

Lemma 2 [1]

"Let x(t) be a solution of (1) and let m(t) = V(t, x(t)), ro = m(0).

Let (4) be satisfied and let $\rho(t)$ be the minimum solution of the equation

$$\dot{r} = \Theta(t, r), \quad r(0) = r_o. \tag{6}$$

Then m(t) $\geq$ $\rho(t)$ as far to the right as both $\rho(t)$ and $\underline{x}(t)$ exists."

The above lemmas are used to verify the following theorems.

## Theorem 1 [1]

"Let $\underline{x}(t)$ be a solution of (1), defined for all $0 \leq t \leq$ to. Then either $\left|\underline{x}(t)\right| \longrightarrow \infty$ as $t \longrightarrow$ to or $\underline{x}(t)$ can be <u>continued</u> beyond to."

## Theorem 2 [1]

"Let (3) be satisfied and suppose all solutions of $\dot{r} = \omega(t, r)$ can be continued for all t. Then all solutions of (1) can be <u>continued</u> for all t."

We now list some corollaries of the above theorems which deal with the problem of boundedness.

## Corollary 1 [1]

"If all the solutions of $\dot{r} = \omega(t, r)$ are <u>bounded</u>, then all the solutions of (1) are bounded."

## Corollary 2 [1]

"If all solutions of $\dot{r} = \omega(t, r)$ approach zero as $t \longrightarrow \infty$, then all the solutions of (1) <u>approach zero</u> as $t \longrightarrow \infty$."

## Corollary 3 [1]

"If the $\lim_{V \to \infty} |\underline{x}| = \infty$ and (4) is satisfied, and if all the solutions of $\dot{r} = \Theta(t, r)$ approach infinity, then all the solutions of (1) <u>approach infinity</u>."

The next two theorems give relationships between the "test functions", $V(t, \underline{x})$, and the solutions, $\underline{x}$.

## Theorem 3 [1]

"Let $\omega$ and $\Theta$ be monotonic nondecreasing in r for all t and suppose that (3) and (4) are satisfied. Let all solutions of $\dot{r} = \omega(t, r)$ and $\dot{r} = \Theta(t, r)$ tend to finite limits as $t \longrightarrow \infty$. Denote $V(0, \underline{x}_0)$ as $r_0$. Let r(t) be the maximum solution of (5) and $\rho(t)$ be the minimum solution of (6). Then if $\underline{x}$ is a solution of (1), $V(t, \underline{x}(t))$ tends toward a finite value as $t \longrightarrow \infty$ for all $\underline{x}$; and $\lim\limits_{t \to \infty} \rho(t) \leqslant$

$$\lim_{t \to \infty} V(t, \underline{x}(t)) \leqslant \lim_{t \to \infty} r(t)."$$

Notice that if $\omega(t, r) \leqslant 0$, then $\dot{V}$ is negative semidefinite and V approaches a limit as $t \to \infty$; but V approaching a limit does <u>not</u> imply that $\underline{x}$ approaches a limit. Brauer in reference [1] gives a counter example to support this statement. The example is as follows:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1$$

where

$$x_1 = a \cos t + b \sin t$$

$$x_2 = -a \sin t + b \cos t \, ;$$

if $V = \underline{x}_T \underline{x}$, then $V = a^2 + b^2$ for all t. Therefore, V approaches a limit as $t \to \infty$, but $\underline{x}$ is periodic and does <u>not</u> approach a limit. The following theorem gives sufficient conditions to guarantee a finite limit for the solutions of (1).

## Theorem 4 [1]

"Let (3) be satisfied and let $\omega$ be monotonic nondecreasing in r for each fixed t. Suppose all solutions of $\dot{r} = \omega(t, r)$ tend to finite limits as $t \longrightarrow \infty$ and

$$V\left(t, \int_a^t \underline{f}(s, \underline{x}(s)) \, ds\right) \leq \int_a^t V(s, \underline{x}(s)) \, ds. \tag{7}$$

Then all solutions of (1) tend to <u>finite limits</u> as $t \longrightarrow \infty$."

Next, we outline some of the results of Rosen, [2]. He obtains a <u>bound</u> for the solutions of (1) and (2), under the following conditions imposed on $\underline{f}$ $(\underline{x}, t)$, by linearizing $\underline{f}$ $(\underline{x}, t)$ and then applying the mean value theorem. We assume that $\underline{f}$ has continuous second partial derivatives. Thus, the mean value theorem yields.

$$\underline{f} \ (\underline{x}, t) = \underline{f} \ (\underline{0}, t) + \underline{B} \ (\gamma_1 \ \underline{x}, t) \ \underline{x} \ ,$$

where $0 \leq \gamma_i \leq 1$ and $\underline{B}$ $(\gamma_i \ \underline{x}, t)$ is the Jacobian matrix $\left\{ \dfrac{\partial f_i(\gamma_i \underline{x}, t)}{\partial x_j} \right\}$ .

Therefore, equation (1) becomes

$$\dot{\underline{x}} = -\underline{B} \ (\gamma_1 \ \underline{x}, t) - \underline{f} \ (\underline{o}, t). \tag{8}$$

## Lemma 3 [2]

"Let $\underline{U}_i = \gamma_i \underline{u}$, $0 \leq \gamma_i \leq 1$ and let $\underline{f}$ $(\underline{u}, t)$ be such that the matrix

$$\underline{B}(\underline{U}_{(i)}, \ t) + \underline{B}_T \ (\underline{U}_{(i)}, \ t)$$

is positive semidefinite for $o \leq t \leq T$. Suppose

$$\left\| \underline{U}_{(i)} \right\| \leq \left\| \underline{x}_0 \right\| + \int_0^T \left\| \underline{f} \ (\underline{o}, t) \right\| \ d \ .$$

Then there exists a unique continuous solution $\underline{x}(t)$ of (1) such that

$$\| \underline{x} \| \leq \| \underline{x}_0 \| + \int_0^T \| \underline{f} \ (\underline{o}, t) \| \ dt.$$"

The hypotheses of Lemma 3 can be weakened so that the existence of a solution depends on the minimum eigenvalue of $\underline{B} + \underline{B}_T$. Hence $\underline{B} + \underline{B}_T$ need <u>not</u> be positive semidefinite. We denote this minimum eigenvalue, over the finite interval $o \leq t \leq T$, by $2 \Lambda \ (\rho)$, where $\rho = \| \underline{u} \|$ .

In general, the matrix $\underline{B}$ cannot be determined since the $\gamma_i$ are unknown. Hence, Rosen, [2], considers the evaluation of $\Lambda(\rho)$ a problem in non-linear programming; and in an appendix of his paper, he shows how the problem can be formulated as a minimization of the variable $\lambda$ in a space of $(2n + \nu + 1)$ dimensions, where $\nu$ is the number of parameters.

In theorem 5, an upper bound for $\|\underline{x}\|$ is given for the interval $o \leq t \leq T$. The bounding function is defined in the following way:

$$\rho(\lambda, t) \equiv \|\underline{x}_0\| e^{-\lambda t} + \int_0^t \|\underline{f}(\underline{o}, \tau)\| e^{-\lambda(t-\tau)} d\tau \geq o$$

and

$$\rho(\lambda) = \underset{[o, T]}{\text{Max}} \rho(\lambda, t) \geq 0.$$

We now consider the equation

$$\Lambda \{\rho(\lambda)\} - \lambda = 0. \tag{9}$$

If the function $\Lambda\{\rho(\lambda)\}$ is bounded and nondecreasing, hence there exists a maximum root $\lambda_m \leq \Lambda(o)$, provided there is at least one $\lambda$ such that

$$\lambda \leq \Lambda\{\rho(\lambda)\} \leq \Lambda(o).$$

### Theorem 5 [2]

"Assume that there exists a root $\lambda_m$ for (9). Then there exists a unique, continuous solution $\underline{x}$ of (1) for which

$$\|\underline{x}\| \leq \rho(\lambda_m, t), \quad o \leq t \leq T."$$

### Example [2]

Consider the first order system

$$\dot{x} = \left\{\frac{x}{1-x}\right\}, \quad 0 < x_0 < 1.$$

Then $f(x, t) = f(x) = 1 - \frac{1}{1-x}$, where $B = \frac{-1}{(1-x)^2}$. Hence, the minimum eigenvalue of $1/2\{\underline{B} + \underline{B}_T\}$ is $\Lambda(\rho) = \frac{-1}{(1-\rho)^2}$. The function $\rho(\lambda, t)$ becomes

$$\rho(\lambda, t) = x_0 e^{-\lambda t}, \quad \rho(\lambda) = x_0 e^{-\lambda T}.$$

**Equation (9) becomes**

$$\frac{1}{\left\{1 - x_0 e^{-\lambda T}\right\}^2} - \lambda = 0.$$

If $x_0 = \frac{1}{10}$ and $T = 1/2$, then $\lambda_m \doteq -1.709$ and $\|\underline{x}\| \leq \rho(\lambda_m, t) = x_0 e^{-\lambda_m t} \leq$ 0.26. From the exact solution and for $x_0 = \frac{1}{10}$ and $T = 1/2$; we have $\|\underline{x}\| \leq 0.18$. Thus, $\rho(\lambda_m, t)$ gives an upper bound.

From theorem 5, we can conclude the following about the asymptotic behavior of the solutions of (1) as $t \longrightarrow \infty$: "if $\|\underline{f}(\underline{0}, t)\|$ is bounded and $\lambda_m > 0$ for $0 \leq t \leq T$ as $T \longrightarrow \infty$, then $\|\underline{x}\|$ is bounded as $t \longrightarrow \infty$."

### Corollary 4 [2]

"Let $\|\underline{f}(\underline{0}, t)\|$ be bounded for $t \geq 0$ and let $\Lambda(\rho)$ and $\|\underline{f}\|$ max. be defined for $T = \infty$. If $\lambda_m > 0$ is the maximum root of the equation

$$\Lambda\left[\|\underline{x}_0\| + \lambda^{-1} \|\underline{f}\|_{max}\right] - \lambda = 0,$$

then there exists a unique, continuous solution of (1) and

$$\|\underline{x}\| \leq \|\underline{x}_0\| e^{-\lambda_m t} + \int_0^t \|\underline{f}(0, \tau)\| e^{-\lambda_m(t - \tau)} d\tau \leq$$

$$\leq \|\underline{x}_0\| + \lambda_m^{-1} \|\underline{f}\| \text{ max., for } t \geq 0."$$

### Example [2]

Consider the second order example given by:

$$\dot{x}_1 = -x_1 x_2 \cos \omega t - 2x_1(1 + \frac{1}{2} x_1)^{-1} + \frac{e^{-t}}{8}$$

$$= -f_1(x_1, x_2, t) ;$$

$$\dot{x}_2 = -(2 - x_1 \cos^2 \omega t) x_2 - (1 + x_2)^{3/2} + 1$$

$$= -f_2(x_1, x_2, t).$$

The matrix $\underline{B}$ is defined as:

$$\underline{B} = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

, $a = \gamma_1 x_2 \cos\phi + 2\left(1 + \frac{1}{2}\gamma_1 x_1\right)^{-2}$

$b = \gamma_1 x_1 \cos\phi$

$c = -\gamma_2 x_2 \cos^2\phi$

$d = 2 - \gamma_2 x_1 \cos^2\phi + \frac{3}{2}\sqrt{1 + \gamma_2 x_2}$

$\phi = \omega t.$

The value for $\Lambda[\rho]$ for any fixed $\rho$ is given by

$$2\Lambda[\rho] = \min_{\substack{0 \le \gamma_1, \gamma_2 \le 1 \\ 0 \le \phi \le 2\pi}} \left[ a + d - \sqrt{(a-d)^2 + (b+c)^2} \right].$$

Thus, for $\rho^2 \le 1$, $\Lambda[\rho] \ge 2(1-\rho)$. Also, $f_1(0, 0, t) = -e^{-t}/8$ and $f_2(0, 0, t) = 0$. Therefore, $\|\underline{f}(0, t)\| = \frac{1}{8}e^{-t}$ and $\|\underline{f}\|_{max} = 1/8$. Hence, $\rho = \|x_0\| + \lambda^{-1}\|\underline{f}\|_{max}$ implies that the maximum root of $2\left(1 - \frac{1}{2} - \frac{1}{8\lambda}\right) - \lambda$ is $1/2$. In conclusion, the bound for $\|\underline{x}\|$ for $t \ge 0$ is

$$\|\underline{x}\| \le 1/2\, e^{-t/2} + 1/8 \int_0^t e^{-\tau - 1/2(t-\tau)}\, d\tau$$

$$\le 2(1 - e^{-t/2})\, e^{-t/2} \le 3/4.$$

As $t \longrightarrow \infty$, $\|\underline{x}\| \longrightarrow 0$.

In the special case when (1) is **linear**, Corollary 5 gives better results than Theorem 5.

### Corollary 5 [2]

"Let $\underline{x}$ (t) satisfy $\dot{\underline{x}} = - \underline{B}$ (t) $\underline{x}$ + $\underline{f}$ (t), $\underline{x}$ (0) = $\underline{x}_0$ . Let $\lambda$ (t) be the minimum eigenvalue of 1/2 $\left[ \underline{B} + \underline{B}_T \right]$ . Then

$$\| \underline{x} \| \le \| \underline{x}_0 \| \exp \left[ - \int_0^t \lambda (\tau) \, d\tau \right] + \int_0^t \| \underline{f}(\tau) \| \exp \left[ - \int_\tau^t \lambda (\sigma) \, d\sigma \right] d\tau . ''$$

### Example [2]

Consider the constant coefficient, second order system given by:

$$\dot{x} = y$$
$$\dot{y} = - \epsilon^2 x - 2 \epsilon y .$$

For $\epsilon > 0$, this system is asymptotically stable.

But, the solution can be made arbitrarily large at a finite time. The solution of the system is

$$x = t \, e^{- \epsilon t} ,$$

$$y = (1 - \epsilon t) \, e^{- \epsilon t} ,$$

where

$$\| \underline{x} \| = \sqrt{t^2 + (1 - \epsilon t)^2} \; e^{- \epsilon t} .$$

For any M > 0, let $\epsilon = 1/M\epsilon$ ; therefore for t = Me , $\| \underline{x} \| = M$. The bound given by Corollary 5 is:

$$\| \underline{x} \| = \sqrt{t^2 + (1 - \epsilon t)^2} \; e^{- \epsilon t} \le e^{(1 - \epsilon)^2 \, t/2} .$$

<u>Example</u> [2]

Consider the second order, linear, time-varying system given by:

$$\dot{x} = -2x + e^{2t} y,$$

$$\dot{y} = e^{-2t} x - 2y,$$

where $x(0) = 1$ and $y(0) = 1 + \sqrt{2}$. The solution is given by:

$$x(t) = e^{(\sqrt{2} - 1) t},$$

$$y(t) = (1 + \sqrt{2}) e^{(\sqrt{2} - 3)t}.$$

Thus, the actual value of $\|\underline{x}\|$ is

$$\|\underline{x}\| = \sqrt{1 + (1 + \sqrt{2})^2 e^{-4t}} \quad \exp\left[(\sqrt{2} - 1) t\right].$$

Therefore the solution grows exponentially. The upper bound given by Corollary 5, where the eigenvalues of $\underline{B}$ are 1 and 3, is

$$\|\underline{x}\| \leq \sqrt{1 + (1 + \sqrt{2})^2} \quad \cdot \quad \exp\left[\frac{\sinh 2 t}{2} - 2 t\right].$$

With these examples, we conclude the <u>boundedness part</u> of this section and we now look at the stability problem of nonautonomous systems.

(II)  <u>THEOREMS ON THE STABILITY OF LINEAR,</u>

<u>NONAUTONOMOUS SYSTEMS</u>

Consider the linear system

$$\dot{\underline{x}} = \underline{A}(t) \underline{x}. \tag{10}$$

It is well known that if $\underline{A}$ were a constant matrix, the system would be asymptotically stable if every eigenvalue of $\underline{A}$ had a negative real part. In [5], Zubov shows that the above statement is not, in general, valid for nonautonomous systems. Yet one might expect that the above criterion would apply to system (10) if the elements of $A(t)$ varied "slowly enough".

That this is the case has been shown by Rosenbrock, [3] .

### Theorem 6 [3]

"Let every element $a_{ij}$ of $\underline{A}(t)$ in (10) be differentiable. Suppose that $|a_{ij}| < a < \infty$ and suppose that every eigenvalue $\lambda$ of $\underline{A}$ satisfies

$$Re(\lambda) \leq -\epsilon < 0.$$

Then there exists some $\delta > 0$ which is independent of $\epsilon$ such that if $|\dot{a}_{ij}| \leq \delta$ for all i, j, then the solution $\underline{x} = \underline{0}$ of (10) is asymptotically stable."

The problem still remains of finding the bound on the $|\dot{a}_{ij}|$. Rosenbrock developed a method for the special case in which $\underline{A}(t)$ takes the form

$$\underline{A}(t) = \begin{array}{|c|c|c|c|c|c|}
\hline
0 & 1 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & 1 & 0 & \cdots & 0 \\
\hline
\cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
\hline
0 & 0 & 0 & 0 & \cdots & 1 \\
\hline
-a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_n \\
\hline
\end{array} \quad , \tag{11}$$

where the $a_i$ are time varying functions. If $\underline{A}$ is of the form in (11), system (10) reduces to the n-th order equation:

$$x^{(n)} + a_n x^{(n-1)} + \cdots + a_1 x = 0.$$

The results of Rosenbrock's work are given in Theorem 7.

### Theorem 7 [3]

"Let $\underline{A}(t)$ be given by (11) and let the hypothesis of Theorem 6 be satisfied. Further, let $|\lambda_i - \lambda_j| \geq \Theta > 0$ for all i, j, i $\neq$ j. Let $|a_i| < a$ for all i. Then the point $\underline{x} = \underline{0}$ is <u>asymptotically stable</u> if the matrix $\underline{L}(t)$ is negative definite for all t $\geq$ to and for some $\eta > 0$, where

$$\underline{L} = \underline{S} \underline{A}_T + \underline{A} \underline{S} - \dot{\underline{S}} + \eta \underline{I} , \tag{12}$$

and

$$(S_{ij}) = \left( \sum_{K=1}^{n} \lambda_K^{i-1} \overline{\lambda_K^{j-1}} \right) . \tag{13}$$

Example [3]

Consider the system

$$\dot{y} + y = \ddot{x} + 2\dot{x} ,$$

$$y = - K(t) \, x ,$$

or

$$\ddot{x} + (2 + K) \, \dot{x} + (K + \dot{K}) \, x = 0. \qquad (14)$$

The eigenvalues of (14) are given by

$$\lambda_1, \lambda_2 = -(2 + K) \pm \frac{\sqrt{4 + K^2 - 4\dot{K}}}{2} .$$

To insure that these are real and satisfy $\left| \lambda_1 - \lambda_2 \right| \geqslant \oslash > 0$, we require that $\left| \dot{K} \right| \leq 1$. Then the matrix $\underline{L}$ is given by

$$\underline{L} = -2 \cdot \begin{array}{|c|c|} \hline 2 + K & 2K - (2 + K)^2 \\ \hline 2K - (2 + K)^2 & (2 + K)^3 - 3K(2 + K) \\ \hline \end{array} +$$

$$- \dot{K} \cdot \begin{array}{|c|c|} \hline 0 & 3 \\ \hline 3 & -2 -4(2 + K) \\ \hline \end{array} - \ddot{K} \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & -2 \\ \hline \end{array} + \eta \, I . \qquad (15)$$

The eigenvalues of the first matrix on the right of (15) are given by:

and

$$\lambda^2 - (2 + K)(5 + K + K^2) \, \lambda + K^3 + 4 \, K = 0$$

$$\min_i \lambda_i \geqslant \frac{K^3 + 4 \, K}{(2 + K)(5 + K + K^2)} . \quad \text{Since} \quad \underline{B} = \underline{B}_T ,$$

$$\min_i \left[ b_{ii} - \sum_{i \neq j} \left| b_{ij} \right| \right] \underline{x}_T \, \underline{x} \leq \min_i \lambda_i(\underline{B}) \, \underline{x}_T \, \underline{x} \leq \underline{x}_T \, \underline{B} \, \underline{x} ,$$

$$\leq \max_i \lambda_i(\underline{B}) \, \underline{x}_T \, \underline{x} ,$$

$$\leq \max_i \left[ b_{ii} + \sum_{i \neq j} \left| b_{ij} \right| \right] \underline{x}_T \, \underline{x} .$$

Hence, $- \left\{ \underline{x}_T \, \underline{L} \, \underline{x} \right\}$ may be estimated by

$$- \underline{x}_T \ \underline{L} \ \underline{x} \ \geqslant \left\{ \frac{2(K^3 + 4 \ K)}{(2 + K)(5 + K + K^4)} - (13 + 4 \ K) \ | \ \dot{K} \ | \ - 2 \ | \ \ddot{K} \ | \ - \eta \right\} \underline{x}_T \ \underline{x};$$

and a sufficient condition for asymptotic stability is:

$$\frac{2(K^3 - 4K)}{(2 + K)(5 + K + K^2)} - (13 + 4 \ K) \ | \ \dot{K} \ | \ - 2 \ | \ \ddot{K} \ | \geqslant \eta > 0.$$

Another criterion for the stability of a homogeneous system is given by Wazewski, [1].

## Theorem 8 [1]

"If $\underline{A}(t)$ is real and continuous and if $\lambda(t)$ is the largest eigenvalue of $\frac{1}{2} (\underline{A} + \underline{A}_T)$, then the solution $\underline{x} = \underline{0}$ of the system (10) is <u>stable</u> if

$$\int_0^\infty \lambda(t) \ dt < \infty \qquad \text{and} \ \underline{\text{asymptotically stable}} \ \text{if} \ \int_0^\infty \lambda(t) \ dt = - \ \infty."$$

In the proof we make use of a V-function of the form: $V(\underline{x}, t) = \underline{x}_T \ \underline{x}$, $\dot{V} \leq 2 \lambda(t) \ \underline{x}_T \ \underline{x}$. From the inequality, $\dot{V} \leq 2 \ (t) \ V$ we obtain an upper bound for $\| \underline{x} \|$, namely:

$$\| \underline{x} \| \leq \| \underline{x}_0 \| \ \exp \left[ \int_0^t \lambda(s) \ ds \right] .$$

From this inequality, the conclusion of Theorem 8 follows.

We now consider some of the stability results of Zubov, [4]. In the following three theorems we use the notation

$r = \| \underline{x} \|$ for a fixed vector $\underline{x}$,

$r_0 = \| \underline{x}_0 \|$,

$r(t, \underline{x}_0, t_0) = \| \underline{x}(t) \|$, where $\underline{x}$ is a solution of (10) for the initial values $\underline{x}(t_0) = \underline{x}_0$.

<u>Theorem 9</u> [4]

"In order that every solution of (10) satisfy the inequalities

$$r_o \ \phi_1^{\frac{1}{2}}(t_o) \ \phi_2^{-\frac{1}{2}}(t) \ \exp\left[-\frac{1}{2}\int_{to}^{t}\left\{\frac{\gamma_1}{\phi_1}\right\}dt\right] \leq r(t, \underline{x}_o, t_o) \leq$$

$$\leq r_o \ \phi_2^{\frac{1}{2}}(t_o) \ \phi_1^{-\frac{1}{2}}(t) \ \exp\left[-\frac{1}{2}\int_{to}^{t}\left\{\frac{\gamma_2}{\phi_2}\right\}dt\right],$$

for $t \geqslant t_o$, it is sufficient that there exist two quadratic forms $V(\underline{x}, t)$ and $W(\underline{x}, t)$ satisfying the following:

(1)  V is nonnegative and $\phi_1(t) \ r^2 \leqslant V \leq \phi_2(t) \ r^2$ ;

(2)  W satisfies $-\gamma_1(t)r^2 \leq W \leq -\gamma_2(t) \ r^2$ ;

(3)  the functions $\phi_1(t) > 0, \ \phi_2(t) > 0, \ \gamma_1(t) \geqslant 0,$

$\gamma_2(t) \geqslant 0$ and $\dfrac{\gamma_1}{\phi_1}$ and $\dfrac{\gamma_2}{\phi_2}$ are integrable;

(4)  $W = \dfrac{dV}{dt}$ ."

<u>Corollary 6</u>

"If the function

$$\phi_1^{-\frac{1}{2}}(t) \ \phi_2^{\frac{1}{2}}(to) \ \exp\left[-1/2\int_{to}^{t}\left\{\frac{\gamma_2}{\phi_2}\right\}dt\right]$$

is bounded from above for all $t \geqslant to > 0$, then the solution $\underline{x} = \underline{0}$ of (10) is <u>stable</u>."

<u>Corollary 7</u>

"If the function
$$\phi_1^{-\frac{1}{2}}(t) \ \phi_2^{\frac{1}{2}}(t_o) \ \exp\left[-\frac{1}{2}\int_{to}^{t}\left\{\frac{\gamma_2}{\phi_2}\right\}dt\right] \quad \text{is}$$

bounded and tends to zero as t ⟶ ∞, then the solution $\underline{x} = \underline{0}$ of (10) is __asymptotically stable__."

## Theorem 10 [4]

"In order that every solution of (10) satisfy the inequalities

$$p_1 \, r_0 \, \phi^{\frac{1}{2}}(t_0) \, \phi^{-\frac{1}{2}}(t) \, \exp\left[ -p_2 \int_{t_0}^{t} \left\{ \frac{\gamma}{\phi} \right\} dt \right] \leq r(t, \underline{x}_0, t_0) \leq$$

$$q_1 \, r_0 \, \phi^{\frac{1}{2}}(t_0) \, \phi^{-\frac{1}{2}}(t) \, \exp\left[ -q_2 \int_{t_0}^{t} \left\{ \frac{\gamma}{\phi} \right\} dt \right]$$

for $t \geqslant t_0$, it is necessary and sufficient that there exist two quadratic forms $V(t, \underline{x})$ and $W(t, \underline{x})$ satisfying the conditions:

(1)  V is nonnegative and satisfies

$a_1 \, \phi(t) \, r^2 \leq V \leq a_2 \, \phi(t) \, r^2$; $a_1, a_2 > 0$.

(2)  W satisfies the conditions

$-b_1 \, \gamma(t) \, r^2 \leq W \leq -b_2 \, \gamma(t) \, r^2$; $b_1, b_2 > 0$;

(3)  $\phi(t) > 0$; $\gamma(t) > 0$ for $t \geqslant 0$; $\frac{\gamma}{\phi}$ is integrable;

(4)  $W = \dot{V}$."

## Corollary 8

"If the conditions of Theorem 10 are satisfied and $\phi(t) = \gamma(t) = 1$, then every solution of (10) satisfies

$$p_1 \, r_0 \, \exp\left[ -p_2 (t - t_0) \right] \leq r(t, \underline{x}_0, t_0) \leq$$

$$\leq q_1 \, r_0 \, \exp\left[ -q_2 (t - t_0) \right],$$

where $p_i$ and $q_i$ are appropriate constants."

## Corollary 9

"If all the coefficients of the system (10) are bounded in absolute value

for $t \geqslant 0$, then in order for the solution $\underline{x} = \underline{0}$ of (10) to be __asymptotically stable__

and that every solution satisfy an estimate of the form

$$r(t, \underline{x}_0, t_0) \leqslant \alpha_1 r_0 \exp\left[- \theta_1 (t - t_0)\right] \quad ,$$

for $t \geqslant t_0$, it is necessary and sufficient that there exist two quadratic forms

V and W satisfying

$$a_1 r^2 \leqslant V \leqslant a_2 r^2 ,$$

$$- b_1 r^2 \leqslant W \leqslant - b_2 r^2 ."$$

## Theorem 11 [4]

"In order that every solution of (10) satisfy the inequalities

$$r_0 \phi_1^{\frac{1}{2}}(t_0) \phi_2^{-\frac{1}{2}}(t) \exp\left[- \frac{1}{2} \int_{t_0}^{t} \left\{\frac{\gamma_1}{\phi_1}\right\} dt\right] \leq r(t, \underline{x}_0, t_0) \leq$$

$$\leq r_0 \phi_2^{\frac{1}{2}}(t_0) \phi_1^{-\frac{1}{2}}(t) \exp\left[- \frac{1}{2} \int_{t_0}^{t} \left\{\frac{\gamma_2}{\phi_2}\right\} dt\right],$$

for $t \geqslant t_0$, it is sufficient that there exist two quadratic forms V and W

satisfying the conditions;

(1) V is nonnegative and satisfies
$$\phi_1(t) r^2 \leq V \leq \phi_2(t) r^2 ;$$

(2) W satisfies
$$-\gamma_1(t) r^2 \leq W \leq - \gamma_2(t) r^2 ;$$

(3) $\phi_1(t) > 0, \phi_2(t) > 0$, for $t \geqslant 0$ and $\dfrac{\gamma_1}{\phi_1}$ , $\dfrac{\gamma_2}{\phi_2}$

are integrable in any finite interval;

(4) $W = \dot{V}$ ;

and where

$$\tilde{\phi}_1 = \sigma_1 \dot{\phi}_1(t) + \sigma_2 \dot{\phi}_2(t) ,$$

$$\tilde{\phi}_2 = \sigma_3 \dot{\phi}_1(t) + \sigma_4 \dot{\phi}_2(t) ,$$

$$\sigma_1 = 1 \text{ for } \gamma_1 \geqslant 0 ,$$

$$\sigma_1 = 0 \text{ for } \gamma_1 < 0 ,$$

$$\sigma_2 = 1 \text{ for } \gamma_1 < 0 ,$$

$$\sigma_2 = 0 \text{ for } \gamma_1 \geqslant 0 ."$$

## Corollary 10

"If the function

$$\gamma (t, t_0) \equiv r_0 \phi_2^{\frac{1}{2}}(t_0) \phi_1^{-\frac{1}{2}}(t) \exp\left[ - 1/2 \int_{t_0}^{t} \frac{\gamma_2}{\tilde{\phi}_2} dt \right]$$

is bounded for all to $\geqslant$ 0 and t $\geqslant$ $t_0$, then the solution $\underline{x}$ = $\underline{0}$ of (10) is __stable__. If in addition $\gamma (t, t_0) \longrightarrow 0$ as t $\longrightarrow$ $\infty$, the solution $\underline{x}$ = $\underline{0}$ will be __asymptotically stable__."

So far we have considered only the homogeneous case; we now turn to the perturbed linear system.

$$\underline{\dot{x}} = \underline{A} (t) \underline{x} + \underline{g} (t, \underline{x}) . \tag{16}$$

If the function $\underline{g}$ (t, $\underline{x}$) is properly restricted, then the asymptotic stability of (16) will follow from the asymptotic stability of (10). We now state some of the possible restrictions on $\underline{g}$ and the corresponding types of stability imposed upon the system.

## Theorem 12 [1]

"Let $\lambda$ (t) be the largest eigenvalue of $\frac{1}{2} \left\{ \underline{A}(t) + \underline{A}_T (t) \right\}$ . Let

$$\lim_{t \to \infty} \left[ \frac{1}{t} \int_{0}^{t} \lambda (s) ds \right] = C < 0$$

and suppose that for all $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$\| \underline{g}(t, \underline{x}) \| < \epsilon \, \| \underline{x} \| \quad \text{for all } \underline{x} \text{ where } \| \underline{x} \| < \delta. \text{ Then the solution}$$

$\underline{x} = \underline{0}$ of (16) is <u>asymptotically stable</u>."

<u>Theorem 13</u>, [16]

"Let $\underline{g}(t, \underline{x})$ satisfy the condition

$$\| \underline{g}(t, \underline{x}) \| \leq h(t) \, \| \underline{x} \|$$

when $\| \underline{x} \| < H \leq \infty$ for all $t \geq 0$. Let $h(t)$ satisfy $\displaystyle\int_0^\infty h(t) \, dt <$

$< \infty$. Then if the system (10) is <u>uniformly stable</u> (i.e., all solutions uniformly

stable), the solution $\underline{x} = \underline{0}$ of (16) is uniformly stable."

<u>Theorem 14</u> [16]

"Let $\underline{g}(t, \underline{x})$ satisfy the modified Lipschitz condition

$$\| \underline{g}(t, \underline{x}_1) - \underline{g}(t, \underline{x}_2) \| \leq h(t) \, \| \underline{x}_1 - \underline{x}_2 \|$$

for all $t \geq 0$ and $\| \underline{x}_1 \| < \infty$, $\| \underline{x}_2 \| < \infty$. Let $h(t)$ satisfy

$\displaystyle\int_0^\infty h(t) \, dt < \infty$. Then, if the system (10) is uniformly stable, the

system (16) is uniformly stable."

Let us now consider the system discussed by Kreidner, [6]:

$$\dot{\underline{x}}(t) = \underline{A}(t) \, \underline{x}(t) + \underline{B}(t) \, \underline{U}(t) ,$$

$$\underline{Y}(t) = \underline{C}(t) \, \underline{x}(t) ,$$

$$\tag{17}$$

where $\underline{x}(t)$ is an n-dimensional state vector, $\underline{U}(t)$ is an R-dimensional input

vector, and $\underline{Y}(t)$ is an m-dimensional output vector. Assume that matrix $\underline{A}(t)$

satisfies a global Lipschitz condition:

$$\| \underline{A}(t) \| \leq K < \infty, \text{ for all } t. \tag{18}$$

We are interested in determining the stability of the system (17) from the stability of the homogeneous equation

$$\dot{\underline{x}}(t) = \underline{A}(t) \underline{x}(t). \tag{19}$$

We know that a unique solution of (19) exists for all t because of condition (18); let this solution be given by:

$$\underline{x}(t) = \bar{\underline{\phi}}(t, t_0) \underline{x}(t_0). \tag{20}$$

Then the solution of (17) is

$$\underline{x}(t) = \bar{\underline{\phi}}(t, t_0) \underline{x}(t_0) + \int_{t_0}^{t} \underline{\phi}(t, \tau) \underline{B}(\tau) \underline{U}(\tau) \, d\tau.$$

If $\underline{x}(t_0) = \underline{0}$, then

$$\underline{Y}(t) = \int_{t_0}^{t} \underline{H}(t, \tau) \underline{U}(\tau) \, d\tau, \tag{21}$$

where $H(t, \tau) = \underline{C}(t) \underline{\phi}(t, \tau) \underline{B}(\tau)$.

Before stating the main theorems of Kreidner we need some definitions. They are as follows:

## Definition 1

"The equilibrium state $\underline{x} = \underline{0}$ of (19) is <u>exponentially stable</u> if the solutions (20) satisfy $\|\underline{x}(t)\| \leq C_1 \cdot \|\underline{x}(t_0)\| e^{-C_2(t-t_0)}$ for every to and for $t \geq$ to, where $C_1$ and $C_2$ are positive constants independent of to."

## Definition 2

"An unexcited (i.e., $\underline{x}(t_0) = \underline{0}$) linear system is <u>output stable</u> if every uniformly bounded input, $\|\underline{U}(t)\| < C_3 < \infty$, produces a uniformly bounded output $\|\underline{Y}(t)\| \leq C_4 < \infty$ for every to and all $t \geq$ to."

The results of Kreidner are now given in theorem - form.

## Theorem 15 [6]

"Let $\|\underline{B}(t)\| \leq c_5 < \infty$ and $\|C(t)\| \leq c_6 < \infty$ for every to and all t $\geqslant$ to. Then exponential stability of (19) implies output stability of (21)."

## Theorem 16 [6]

"Let $\underline{B}(t)$ be such that each row has at least one element $b_{ij}(t)$ satisfying $|b_{ij}(t)| \geqslant c_7 > 0$ for some to and all t $\geqslant$ to. Suppose that every column of $\underline{C}(t)$ has at least one element $C_{ij}(t)$ satisfying $|C_{ij}(t)| \geqslant c_8 > 0$ for some to and all t $\geqslant$ to. Suppose further that the system (17) is __completely__ __controllable__ and __completely observable__, that is, respectively,

$$\underline{B}_T(\tau)\ \underline{\Phi}_T(to,\tau)\ \underline{\lambda} \not\equiv \underline{0} \text{ for some to, all } \tau \geqslant to, \text{ and every } \underline{\lambda} \not\equiv \underline{0};$$

and $\underline{C}(t)\underline{\Phi}(t, to) \underline{\lambda} \not\equiv \underline{0}$ for some to, all t $\geqslant$ to and every $\underline{\lambda} \not\equiv \underline{0}$ in state space. Then, output stability implies exponential stability."

By strengthening the hypotheses slightly, theorems 15 and 16 can be combined into a more compact form; that is:

## Theorem 17 [6]

"Exponential stability and output stability of a linear system (17) are equivalent if the following hold:

(1) the system is completely controllable and completely observable;

(2) $0 < \beta_1 \leq \|\underline{B}_T(t) \underline{\lambda}\| \leq \beta_2 < \infty$ for all t and every finite $\underline{\lambda} \not\equiv \underline{0}$ in the state space;

(3) $0 < \gamma_1 \leq \|\underline{C}(t) \underline{\lambda}\| \leq \gamma_2 < \infty$ for all t and every finite $\underline{\lambda} \not\equiv \underline{0}$ in the state space."

(III) THEOREMS ON THE STABILITY OF NONLINEAR, NONAUTONOMOUS SYSTEMS

We begin our discussion of the nonlinear case by considering the system of "first approximation." Consider the system

$$\dot{\underline{x}} = \underline{P}(t) \, \underline{x} + \underline{g} \, (\underline{x}, \, t) \tag{22}$$

where the components of $\underline{g}$ $(\underline{x}, t)$ one analytic functions of $\underline{x}$ for all $t \geqslant 0$ and all $\underline{x}$ such that $\|\underline{x}\| \leq A, A > 0$. Assume that the power series expansion of $\underline{g}$ in $\underline{x}$ begins with terms of the second degree or higher. In this case we call the system

$$\dot{\underline{x}} = \underline{P}(t) \, \underline{x}, \tag{23}$$

the system of first approximation. We now state a few definitions.

The first definition which we consider is what Liapunov has called the characteristic number of f(t). (The following discussion is taken from reference [17]. The function f(t) is bounded for $t > 0$ if $|f(t)| < A$ for sufficiently large A, and unbounded if $|f(t)| > A$ for some t, no matter how large A may be. A function f(t) is called vanishing if $\lim_{t \to \infty} f(t) = 0$. It can be shown if there exists two numbers a and b such that $f(t) e^{at}$ is unbounded and $f(t) e^{bt}$ is vanishing, then there is a real $\lambda$ such that, for any $\epsilon > 0$, $f(t) e^{(\lambda + \epsilon)t}$ is unbounded and $f(t) e^{(\lambda - \epsilon)t}$ is vanishing.

Definition 3 [17]

"Liapunov has called the above number $\lambda$, the characteristic number of f(t). One can also show that

$$\lambda = - \overline{\lim_{t \to \infty}} \, \frac{\ln |f(t)|}{t} \, . \text{''}$$

Some important properties of characteristic numbers are now listed.

(1) The characteristic number of a sum of two functions is the smaller of the two characteristic numbers if the latter are different; it may be smaller than either when they are equal.

(2)  The characteristic number of a product of two functions is not less than the sum of their characteristic numbers.

(3)  The sum of the characteristic numbers of f and 1/f is not greater than zero, it is zero if and only if $( \ln |f(t)| )/t$ approaches a finite limit as $t \longrightarrow \infty$.

(4)  The characteristic number of a product of f and some function g is equal to the sum of their characteristic numbers if the necessary and sufficient condition in (3) is satisfied.

(5)  The characteristic number of an integral is not less than that of the integrand.

(6)  Every nonzero solution of (22), where the $P_{ij}(t)$ are finite for $t \geqslant 0$ and are continuous, has a finite characteristic number.

## Definition 4  [7]

"A system of n independent solutions of a system of differential equations is <u>normal</u> if the sum of the characteristic numbers of all remaining independent solutions attains its supremum."

## Definition 5  [7]

"A system of differential equations is <u>regular</u> if the sum of the characteristic numbers of the normal system of its independent solutions is equal to the negative of the characteristic number of the function

$$\exp \left\{ - \int \sum_{s} P_{ss}(t) \, dt \right\} \qquad \text{where } \underline{P}(t) \text{ in (22)}$$

is expressed as  $\left[ P_{ij}(t) \right]$ ."

Consider the case where the system of first approximation is <u>regular</u>. The following theorems deal with the stability of the undisturbed motion of (22).

<u>Theorem 18</u>  (Liapunov)

"If the system (23) is regular and all of its characteristic numbers are positive, then the undisturbed motion of (22) is asymptotically stable.

<u>Theorem 19</u>  [7]

"If the system (23) is regular and if among its characteristic numbers there exists at least one negative value, then the undisturbed motion of (22) is <u>unstable</u>."

Now consider the case where the system of the first approximation is <u>not regular</u>. Denote by $\Delta$ the determinant constructed from the functions $x_{ij}(t)$, where $x_{1r}, \ldots, x_{nr}, r = 1, \ldots, n$ are the components of the <u>normal</u> set of solutions of (23). Denote by $\Delta_{ij}$ the cofactor of $x_{ij}$ in $\Delta$. Denote by S the sum of the characteristic numbers of the normal set of solutions and let $\mu$ be the characteristic number of the function $1/\Delta$. Call $\sigma = - S - \mu$.

<u>Theorem 20</u>  (Liapunov)

"If the system (23) is not regular and if each of its characteristic numbers is greater than $\sigma$, then the undisturbed motion of (22) is <u>asymptotically stable</u>."

<u>Theorem 21</u>  [7]

"If the system is not regular and the smallest characteristic number is less than $-\sigma$, then the undisturbed motion of the system (22) is <u>unstable</u>."

We now turn to a more general nonlinear, nonautonomous system given by

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}).$$ 

(24)

We require that the components, fs, of $\underline{f}$ be real functions in some region (h):

$t \geqslant 0$, $\|\underline{x}\| \leq R_o$. We also require that $f_s(t, \underline{0}) = 0$ for all $s$; $f_s$ be

continuous in t, and that $f_s$ satisfy the Cauchy condition relative to $x_1$ , . . . . , $x_n$. (The Cauchy condition guarantees the existence of a solution of (24).)

### Definition 6 [8]

"Suppose that in some region (g): t $\geqslant$ 0, $\|\underline{x}\| \leqslant r$ (r $\leqslant R_0$), there is defined a continuous function V(t, $\underline{x}$) which is of definite sign and positive for any fixed t $\geqslant$ 0. Suppose there exists some real constant a $>$ 0 (a $<$ r) such that for every initial value to $\geqslant$ 0 and for every given $\epsilon > 0$ there exists a value of t = T($\epsilon$, to) such that in the plane t = T of the region(g) one can always connect the point $\Theta$(T, $\underline{O}$) with the surface $\|\underline{x}\|$ = a by means of a continuous curve $\Gamma$ at all of whose points V(T, $\underline{x}$) < $\epsilon$. Then, we shall say that the function V is positive weakly definite."

### Definite 7 [8]

"Let V(t, $\underline{x}$) be positive definite in (g) and let $\eta > 0$ be an arbitrary given number. Denote by D($\eta$) the set of all those points of (g) at which V(t, $\underline{x}$) $\leqslant \eta$. Let t = t' $\geqslant$ 0. The intersection of the set D($\eta$) with the plane t = t' in (g) we shall call $\sigma$ ($\eta$, t'). If for every sufficiently small $\eta > 0$, the maximum norm of the points (t', $\underline{x}$) $\in \sigma$ ($\eta$, t') satisfies the condition

$$\text{Max } \|\underline{x}\| \longrightarrow 0 \text{ as } t' \longrightarrow \infty ,$$

then we say the function V(t, $\underline{x}$) is positive strongly definite."

It can be shown, [8], that if V (t, $\underline{x}$) is the positive definite quadratic form

$$V = \underline{x}_T \underline{A}(t) \underline{x}$$

and $\lambda_1$, is the smallest eigenvalue of $\underline{A}$(t), then V is positive strongly definite if and only if $\lim_{t \longrightarrow \infty} \lambda_1(t) = \infty$ and V is positively weakly definite if and

only if $\lim\limits_{t \longrightarrow \infty} \lambda_1(t) = 0$.

We now return to the problem of the stability of equation (24). The next seven theorems will summarize some of the <u>results of Persidskii and Zubov</u>.

<u>Theorem 22</u> [8]

"In order that the solution $\underline{x} = \underline{0}$ of (24) be <u>unstable</u>, it is necessary and sufficient that in some region (g): $t \geqslant 0$, $\|\underline{x}\| \leqslant r \leqslant R_0$, there exists a positive weakly definite function $V(t, \underline{x})$ such that $\dot{V} \geqslant 0$ on the basis of equation (24)."

<u>Example</u> [8]

Consider the third order system given by

$$\dot{x} = - 2xyz ,$$

$$\dot{y} = \left\{ \frac{y}{t + 2} + x^2 z \right\} ,$$

$$\dot{z} = \frac{z}{t + 2} + x^2 y .$$

Let V be defined by the following:

$$V(t, x, y, z) = x^2 + y^2 + z^2 + 2 \left\{ \frac{t + 1}{t + 2} \right\} z y .$$

This V - function is positive weakly definite and the corresponding $\dot{V}$ is:

$$\dot{V} = \frac{2}{t + 2} \left\{ z^2 + y^2 + \left[ \frac{2t + 3}{t + 2} \right] z y + x^2 (t + 1)(z^2 + y^2) \right\} ,$$

$$\dot{V} \geqslant 0.$$

Hence, the zero solution of the above system is <u>not stable</u>.

<u>Theorem 23</u> [8]

"In order that the solution $\underline{x} = \underline{0}$ of the system (24) be <u>asymptotically uniformly stable</u> with respect to the coordinates of $\underline{x}_0$, it is necessary and sufficient that in some region (g): $t \geqslant 0$, $\|\underline{x}\| \leqslant r \leqslant R_0$, there exists a

positive strongly definite function $V(t, \underline{x})$ such that $\dot{V}(t, \underline{x}) \leq 0$ in view of the system (24)."

For the next theorem, Zubov [5] requires that $\underline{f}$ $(t, \underline{x})$ be defined for all $t \geqslant 0$ and all $\underline{x}$ in $E_n$. Also, it is assumed that $\underline{x} = \underline{0}$ is a solution of (24).

Theorem 24    [5]

"In order that the solution $\underline{x} = \underline{0}$ of (24) be stable it is necessary and sufficient that there exists a function $V(t, \underline{x})$ defined for $\|\underline{x}\| < r$, $t \geqslant 0$ with the following properties:

(1)   V is positive definite;

(2)   $V(\underline{x}, t) \longrightarrow 0$ as $\|\underline{x}\| \longrightarrow 0$ uniformly with respect to t;

(3)   the function $V(t, \underline{x}(t, t_0, \underline{x}_0))$ does not increase when $t \geqslant t_0$

where $\|\underline{x}(t, t_0, \underline{x}_0)\| \leq r$.

If furthermore, $V \longrightarrow 0$ as $t \longrightarrow +\infty$ when $\|\underline{x}_0\| < \delta$, then the solution $\underline{x} = \underline{0}$ is asymptotically stable."

Now, Zubov requires that system (24) have a solution $\underline{x} = \underline{x}(t, t_0, \underline{x}_0)$ for any finite $(\underline{x}_0, t_0)$ belonging to the region $t_0 \geqslant 0$, $\|\underline{x}_0\| < H$. Assume, also that $\underline{f}(t, \underline{0}) \equiv \underline{0}$ for all $t > 0$. Consider two continuous positive functions $\phi(t)$ and $\gamma(t)$ defined for all $t \geqslant 0$.
Assume that

$$\gamma(t, t, k_1) = {}^{k+1}\!\!\sqrt{\phi(t_0)\, \phi^{-1}(t)}\ \exp\left[-\int_{t_0}^{t} k_1 \gamma \phi^{-1} dt\right] \longrightarrow 0,$$
and
$$\int_{t_0}^{t} \gamma\, \phi^{-1}\, dt \longrightarrow \infty \qquad \text{as } t \longrightarrow \infty.$$

Also, assume that $\gamma$ is bounded for all $t \geqslant t_0 \geqslant 0$, where k and $k_1$ are positive constants.

**Theorems 25** [5]

"In order that the trivial solution of (24) be _asymptotically stable_ and that any solution beginning in a sufficiently small vicinity of the semi-axis $\underline{x} = \underline{0}$, $t \geqslant 0$ satisfy the inequality:

$$L_1 \; \chi(t, t_0, k_1) \; \|\underline{x}_0\|^2 \leq \|\underline{x}(t, t_0, \underline{x}_0)\|^2 \leq L_2 \chi(t, t, k_2) \; \|\underline{x}_0\| ,$$

it is necessary and sufficient that there exist two functionals V and W of the form

$$V = ( \|\underline{x}\|^2 )^k \; \phi(t) \; \underline{x}_T \underline{A}(t) \; \underline{x} ,$$

$$W = - \gamma(t)( \|\underline{x}\|^2 )^k \; \underline{x}_T \underline{B}(t) \; \underline{x} ,$$

and having the following properties

(1)  V and W are defined and continuous when

$$\|\underline{x}\|^2 \leq d_1 , \; t \geqslant 0 \text{ for sufficiently small } d_1;$$

(2)  $a_1 \|\underline{x}\|^2 \leq \underline{x}_T \underline{A}(t) \; \underline{x} \leq a_2 \|\underline{x}\|^2$

$b_1 \|\underline{x}\|^2 \leq \underline{x}_T \underline{B}(t) \; \underline{x} \leq b_2 \|\underline{x}\|^2$

where $a_1$, $a_2$, $b_1$, $b_2$ are positive constants;

(3)  W is continuous and $\dot{V} = W$."

**Theorem 26** [5]

" Let

$$\chi_2(t, t_0, C) = \frac{\sqrt[k+1]{\phi(t_0) \, \phi^{-1}(t)}}{\left\{ 1 + c \int_{t_0}^{t} \gamma \sqrt[k+1]{\phi^{-(l+1)}(t)} \, dt \right\}^{\frac{1}{l-k}}} ,$$

where k, l and c one positive constants, and $l \gg k$. Assume that $\chi_2 \longrightarrow 0$ as $t \longrightarrow \infty$ and is bounded when $t \geqslant t_0 \geqslant 0$ and $C \geqslant 0$.

Assume that the quantity

$$\Theta = \int_{to}^{t} \left[ \gamma \sqrt[k+1]{\phi^{-(l+1)}} {}_{(t)} \right] dt \longrightarrow + \infty \qquad \text{as } t \longrightarrow + \infty.$$

In order that the trivial solution of (24) be __asymptotically stable__ and in order that the inequality

$$M_1 \, \|\underline{x}_0\|^2 \, \gamma_2(t, to, M_2 \, C) \leq \| \underline{x}(t, to, \underline{x}_0) \|^2 \leq N_2 \, \|\underline{x}_0\|^2 \, \gamma_2(t, to, N_2 \, C),$$

where $\quad C = \left[ ( \|\underline{x}_0\| )^2 \sqrt[k+1]{\phi(to)} \right]^{l-k}$ , be

satisfied for any solution beginning in a sufficiently small vicinity of the semiaxis $\underline{x} = \underline{0}$ , $t \geqslant 0$, it is necessary and sufficient that there exist two functions $V(\underline{x}, t)$ and $W(\underline{x}, t)$ having the following properties:

    (1)  V and W are defined and continuous for $t \geqslant 0$, $\|\underline{x}\|^2 \leq \delta < H$ ;

    (2)  $V = (\|\underline{x}\|^2)^k \, \phi(t) \, \underline{x}_T \underline{A}(t) \, \underline{x}$ ,

        $W = - (\|\underline{x}\|^2)^l \, \gamma(t) \, \underline{x}_T \, \underline{B}(t) \, \underline{x}$ ;

where $\underline{A}$ and $\underline{B}$ are such that there exist positive constants $a_1$, $a_2$, $b_1$, $b_2$ such

$$a_1 \, \|\underline{x}\|^2 \quad \leq \quad \underline{x}_T \, \underline{A}(t) \, \underline{x} \leq a_2 \, \|\underline{x}\|^2$$
$$b_1 \, \|\underline{x}\|^2 \quad \leq \quad \underline{x}_T \, \underline{B}(t) \, \underline{x} \leq b_2 \, \|\underline{x}\|^2 \; ;$$

(3)  $W = \dot{V}.$"

## Theorem 27    [5]

    "Let

$$\gamma_3(t, to, C, l) = \phi(to) \, \phi^{-1}(t) \sqrt[1-l]{1 + C \int_{to}^{t} \gamma \, \phi^{-1} \, dt}$$

where $l > 1$ and $C \geqslant 0$. Assume that $\chi_3$ has the following properties:

(1) $\chi_3 \longrightarrow 0$ as $t \longrightarrow + \infty$ ;

(2) $\chi_3$ is bounded when $t \geqslant t_0 \geqslant 0$ and $C \geqslant 0$;

(3) $\displaystyle \int_{t_0}^{t} \gamma \, \phi^{-1} \, dt \longrightarrow + \infty$ as $t \longrightarrow + \infty$ .

In order for the trivial solution of (24) to be <u>asymptotically stable</u> and in order that any solution beginning in the vicinity of $\| x_0 \| \leq \delta$, $t_0 \geqslant 0$, where $\delta > 0$ is sufficiently small to satisfy the inequality

$$M_1 \, V_1 \, (x_0) \, \chi_3(t, \, t_0, \, M_2 \, C(l), \, l \,) \leq V_1 \, (x(t, \, t_0, \, x_0)) \leq$$

$$\leq N_1 \, V_1 \, (x_0) \, \chi_3(t, \, t_0, \, N_2 C(l), \, l) \, ,$$

where $C(l) = \left[ \phi(t_0) \, V_1(x_0) \right]^{l-1}$ and where $V_1$ is positive definite, it is necessary and sufficient that there exist two functions $V(t, \, x)$ and $W(t, \, x)$ satisfying the conditions:

(1)  V and W are defined and continuous in a sufficiently small vicinity of the semiaxis $t \geqslant 0$, $x = 0$ ;

(2)  $a_1 \, \phi(t) \, V_1 \, (x) \leq V \, (x, \, t) \leq a_2 \, \phi(t) \, V_1 \, (x)$ and $W = -\gamma W_1$ where $W_1$ satisfies

$$b_1 \, V_1^{l_1} \leq W_1 \leq b_2 \, V_1^{l_2} \, ,$$

where $a_1$ , $a_2$ , $b_1$ , $b_2$ are positive;

(3)  $W = \dot{V}$."

## Theorem 28    [5]

"If there exist two functions $V(t, \, x)$ and $W(t, \, x)$ with the following properties:

(1)  V and W are defined and continuous in the region $t \geqslant 0$, $\| x \|^2 < \delta$, where $\delta > 0$ is sufficiently small;

(2) $a_1 V_1(\underline{x}) \leq V(t, \underline{x}) \left[\phi(t)\right]^{-1} \leq a_2 V_1(\underline{x})$,

$$b_1 V_1^{l_1}(\underline{x}) \, \psi(t) \leq - W(t, \underline{x}) \leq b_2 V_1^{l_2}(\underline{x}) \, \psi(t),$$

where $1 < l_1 \leq l_2$;

(3) $W = \dot{V}$,

then the trivial solution of (24) is <u>asymptotically stable</u> and any solution beginning in a sufficiently small vicinity of the semiaxis $\underline{x} = \underline{0}$, $t \geq 0$, satisfies the inequalities

$$V_1(\underline{x}_0) M_1 \, \chi_3(t, t_0, M_2 C(l_1), l_1) \leq V_1(x(t, t_0, \underline{x}_0)) \leq$$

$$\leq N_1 V_1(x_0) \, \chi_3(t, t_0, N_2 C(l_2), l_2),$$

where $C(l) = \left[\phi(t_0) \, V_1(\underline{x}_0)\right]^{l-1}$.

We now consider one theorem dealing with the problem of <u>stability in a finite interval of time</u>. This work is due to Kamenkov and Lebedev and is reported in reference [4], by Zubov.

<u>Definition 8</u>  [4]

"The homogeneous solution of system (24) is called <u>stable for a given to</u> in relation to a positive definite function $V(\underline{x}, t)$ <u>in the interval $\tau$</u> if from $V_1(\underline{x}_0, t_0) = a$ it follows that $V_1(t, \underline{x}_0, t_0) \leq a$ for $(to \leq t \leq t_0 + \tau)$ for every sufficiently small positive a, where $V_1(t, \underline{x}_0, t_0)$ denotes the value $V(\underline{x}, t)$ on an integral curve passing through $\underline{x}_0$ at $t = t_0$."

<u>Definition 9</u>  [4]

"The homogeneous solution of the system (24) is <u>stable in a finite interval</u> for a given to, if there exists a positive constant $\tau$ and a positive definite function $V(\underline{x}, t)$ such that in relation to V the homogeneous solution of (24)

is stable in the finite interval $\tau$".

Consider the system

$$\dot{\underline{x}} = \underline{P}(t)\,\underline{x} + \underline{f}(t, \underline{x}) \tag{25}$$

where $\underline{P}$ is real and continuous for $t \geqslant 0$ and the components $f_s$ of $\underline{f}$ satisfy

$$|f_s| \leqslant \phi_s(t) \left\{ \sum_{i=1}^{n} x_i^2 \right\}^C \tag{26}$$

where $C > 1/2$, and $\phi_s$ are continuous and positive.

Theorem 29    [4]

"In order that the homogeneous solution of (25) be stable in a finite interval for a given to with respect to some fixed quadratic form V (V is positive definite), it is necessary and sufficient that $f_s$ satisfy (26) and the eigenvalues of $\underline{P}(t_0)$ have negative real parts."

IV  METHODS FOR CONSTRUCTING LIAPUNOV FUNCTIONS

In the first method we transform the system of equations into normal coördinates and then consider a Liapunov function which is the sum of the squares of the normal coordinates.  Bulgakov [11] studied this transformation in detail, but the work is in Russian and thus we will consider the discussion of this transformation which is due to Roitenberg, reference [9].

Consider the system of linear differential equations

$$\sum_{K=1}^{n} \phi_{jK}(D)\, x_K = 0 \quad (j = 1, \ldots, n) \tag{27}$$

where $\phi_{jK}(D) = b_{jK}^{(o)}(t)\, D^L + b_{jK}^{(1)}(t)\, D^{L-1} + \ldots + b_{K}^{(L-1)}(t)\, D + b_{jK}^{(L)}(t)$,

$D = \dfrac{d}{dt}$,    $b_{jK}^{(s)}(t) = b_{jK}^{(s)}(t) - a_{jK}^{(s)}$,    and

where $a_{jK}^{(s)}$ are constants. Then we rewrite $\phi_{jK}(D)$ as

$$\phi_{jK}(D) = f_{jK}(D) + L_{jK}(D) ,$$

$$f_{jK}(D) = a_{jK}^{(0)} D^L + \ldots + a_{jK}^{(L-1)} D + a_{jK}^{(L)} ,$$

$$L_{jK}(D) = l_{jK}^{(0)}(t) D^L + \ldots + l_{jK}^{(L-1)}(t) D + l_{jK}^{(L)}(t).$$

The system in (27) becomes

$$\sum_{K=1}^{n} f_{jK}(D) x_K = - \sum_{K=1}^{n} L_{jK}(D) x_K \quad (j = 1, \ldots, n). \tag{28}$$

Notice that the $f_{jK}(D)$ are polynomials in D with constant coefficients. Consider the eigenvalues of the operator matrix $[f_{jK}(D)]$; denote the real eigenvalues by $K_g$ (g = 1, ..., N') and the conjugate complex eigenvalues by $(\epsilon_h \pm i \omega_h)$ (h = N' + 1, ..., N' + N''). We assume that the eigenvalues are distinct.

We now transform the system (28) from the original coordinates, $x_j$, to the normal coordinates, $\xi_g, \zeta_h, \eta_h$ (g = 1, ... , N' ; h = N' + 1, ... , N' + N''). The formulae relating the coordinates are as follows:

$$x_j^{(\Theta)} = \sum_{g=1}^{N'} \left[ \bar{x}_{j\Theta}^{(g)} \xi_g \right] + \sum_{h=N'+1}^{N'+N''} (\bar{x}_{j\Theta}^{(h)} \xi_h + Y_{j\Theta}^{(h)} \eta_h) , \quad \left\{ \begin{array}{l} j = 1, \ldots, n \\ \Theta = 0, 1, \ldots, m_j - 1 \end{array} \right\}$$

$$x_{j\Theta}^{(g)} = \bar{x}_j^{(g)} \chi_g^{(\Theta)} , \quad \bar{x}_{j\Theta}^{(h)} = N_{j\Theta}^{(h)} \cos (\gamma_j^{(h)} + \Theta \zeta_h) ,$$

$$\bar{Y}_{j\Theta}^{(h)} = N_{j\Theta}^{(h)} \sin (\gamma_j^{(h)} + \Theta \zeta_h) ,$$

$$N_{j\Theta}^{(h)} = N_j^{(h)} c_h^{\Theta} ,$$

where $c_h$ and $\zeta_h$ are determined by

$$\epsilon_h + i \omega_h = c_h e^{i \zeta_h} .$$

The quantity $\bar{X}_j^{(g)}$ is the j th element of the non-zero column matrix $\bar{\bar{X}}_g$ of the adjoined matrix $\underline{F}(K_g)$ constructed for the real root $K_g$. The quantity $N_j^{(g)} e^{i \gamma_j^{(h)}} =$ $\bar{X}_j^{(h)} + i Y_j^{(h)}$ is the $j\underline{th}$ element of the non-zero column matrix $\bar{\bar{X}}_h$ of the adjoined matrix $\underline{F}(\epsilon_h + i \omega_h)$ constructed for the complex root $\epsilon_h + i \omega_h$. The quantity $\gamma$ is the order of the highest derivative of $x_j$ on the left-hand side of equation (28).

The right hand side of (28) transforms to:

$$\Lambda_j(\xi_g, \xi_h, \eta_h, t) \equiv - \sum_{K=1}^{n} L_{jK}(D) x_K =$$

$$= \sum_{g=1}^{N'} \mathcal{M}_{jg}(t) \xi_g + \sum_{h=N'+1}^{N'+N'''} \left[ \mathcal{M}_{jh}(t) \xi_h + \mathcal{V}_{jK}(t) \eta_h \right] .$$

Finally, the system in normal coordinates is given by:

$$\dot{\xi}_g = \chi_g \xi_4 + \left[ \frac{D - \chi_g}{\Delta(D)} \right] \cdot \left[ \sum_{K=1}^{n} B_K \overset{(g)}{\Lambda}_K \right]_{D = \chi_g}$$

$$(g = 1, \ldots, N')$$

$$\dot{\xi}_h = \epsilon_h \xi_h + \omega_h \eta_h + 2 \text{ Re} \left[ \frac{D - \epsilon_h - i \omega_h}{\Delta(D)} \right] \cdot \left[ \sum_{K=1}^{n} B_k^{(h)} \Lambda_k \right]_{D = \epsilon_h + i \omega_h}$$

$$\dot{\eta}_h = \epsilon_h \eta_h - \omega_h \xi_h - 2 \text{ Im} \left[ \frac{D - \epsilon_h - i \omega_h}{\Delta(D)} \right] \left[ \sum_{K=1}^{n} B_K^{(h)} \Lambda_K \right]_{D = \epsilon_h + i \omega_h} , \quad (29)$$

$$(h = N' + 1, \ldots, N' + N''),$$

where $\Delta(D)$ is the determinant of the operator matrix $\left[ f_{jK}(D) \right]$.

We then choose as a Liapunov function the following:

$$V = - 1/2 \left\{ \sum_{g=1}^{N'} \xi_g^2 + \sum_{h=N'+1}^{N'+N''} (\xi_h^2 + \eta_h^2) \right\} .$$

Since V is always negative definite, the stability of the system can be examined

by considering $\dot{V}$ with respect to system (29).

## Example [9]

Consider the system of differential equations:

$$ax_1 + \dot{x}_2 = 0,$$

$$\dot{x}_1 - b/a\ x_2 - bk/a\ x_3 + bk/a\ x_4 = 0, \tag{30}$$

$$- M(t)\ x_1 + cx_2 + \dot{x}_3 + cx_3 = 0,$$

$$\dot{x}_4 + c\ x_4 = 0,$$

where M(t) is bounded and we assume that $|M(t)| \leq$ fa. Thus, the third

equation in (30) becomes.

$$- fax_1 + cx_2 + \dot{x}_3 + cx_3 = S(t)x_1,$$

where $S(t) = M(t) - fa$. Then $- 2fa \leq S(t) \leq 0$. With this modification,

(30) corresponds to system (28). The eigenvalues of the matrix $\left[f_{jK}(D)\right]$

are the solutions of $(D + C) \left[D^3 + CD^2 + b(1 - fk)\ D + (1 - k)bc\right] = 0$.

Assume that this equation has two real roots and one pair of complex-conjugate

complex roots, denoted by $K_1, K_2, \in \pm i\omega$; and let $K_1 = - C$. Thus the

transformed system in the normal coordinates $\xi_1, \xi_2, \xi_3, \eta_3$ is given by:

$$\dot{\xi}_1 = K_1 \xi_1,$$

$$\dot{\xi}_2 = \left[K_2 + a_2\ S(t)\right]\xi_2 + a_3\ S(t)\ \xi_3 + a_4\ S(t)\eta_3,$$

$$\dot{\xi}_3 = b_2\ S(t)\ \xi_2 + \left[\in + b_3\ S(t)\right]\ \xi_3 + \left[\omega + b_4\ S(t)\right]\eta_3,$$

$$\dot{\eta}_3 = c_2\ S(t)\ \xi_2 + \left[-\omega + c_3\ S(t)\right]\xi_3 + \left[\in + c_4\ S(t)\right]\eta_3,$$

where the equations of transformation are

$$x_1 = \frac{K_2(K_2 + C)}{(f K_2 + C)\ a}\ \xi_2 + M_1\ \xi_3 + M_2\ \eta_3,$$

$$x_2 = -\frac{K_2 + C}{f K_2 + C}\ \xi_2 + N_1\ \xi_3 + N_2\ \eta_3,$$

$$x_3 = \dot{\xi}_1 + \dot{\xi}_2 + \dot{\xi}_3, \quad x_4 = \dot{\xi}_1,$$

and where

$$M_1 = \frac{1}{a\ell}\left\{ f\epsilon^3 + c(1+f)\epsilon^2 + (c^2 + f\omega^2)\epsilon - c\omega^2(1-f) \right\},$$

$$M_2 = \frac{\omega}{a\ell}\left\{ f\epsilon^3 + 2c\epsilon + c^2 + f\omega^2 \right\},$$

$$N_1 = -\frac{1}{\ell}\left\{ f\epsilon^2 + c(1+f)\epsilon + c^2 + f\omega^2 \right\},$$

$$N_2 = -\frac{\omega c}{\ell}\left\{ 1-f \right\}, \quad \ell = f^2\epsilon^2 + 2cf\epsilon + c^2 + f^2\omega^2,$$

$$a_2 = -A_2\left\{ \frac{\chi_2(\chi_2 + c)}{(f\chi_2 + c)a} \right\}, \quad a_3 = A_2M_1, \quad a_4 = A_2M_2,$$

$$b_2 = A_3\left\{ \frac{\chi_2(\chi_2 + c)}{(f\chi_2 + c)a} \right\}, \quad b_3 = A_3M_1, \quad b_4 = A_3M_2,$$

$$c_2 = A_4\left\{ \frac{\chi_2(\chi_2 + c)}{(f\chi_2 + c)a} \right\}, \quad c_3 = A_4M_1, \quad c_4 = A_4M_2,$$

$$A_2 = T_2(f\chi_2 + c)bk, \quad A_3 = \left[ (f\epsilon + c)T_3 + f\omega T_4 \right]bk,$$

$$A_4 = \left[ (f\epsilon + c)T_4 - f\omega T_3 \right]bk,$$

$$T_2 = \left\{ \frac{1}{(\chi_2 + c)\left[(\chi_2 - \epsilon)^2 + \omega^2\right]} \right\},$$

$$T_3 = \frac{-(2\epsilon - \chi_2 + c)}{\left[(\epsilon + c)(\epsilon - \chi_2) - \omega^2\right]^2 + \left[(2\epsilon - \chi_2 + c)\omega\right]^2},$$

$$T_4 = \frac{(\epsilon + c)(\epsilon - \chi_2) - \omega^2}{\omega\left\{\left[(\epsilon + c)(\epsilon - \chi_2) - \omega^2\right]^2 + \left[(2\epsilon - \chi_2 + c)\omega\right]^2\right\}}.$$

Select the following Liapunov function:

$$V = -1/2\left\{ \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2 + \eta_3^2 \right\},$$

where

$$\dot{V} = (\xi_1, \xi_2, \xi_3, \eta_3)\ \underline{A}\ (\xi_1, \xi_2, \xi_3, \eta_3)_T\ ,$$

$$\underline{A} = \begin{array}{|c|c|c|c|} \hline -\varkappa_1 & 0 & 0 & 0 \\ \hline 0 & -\varkappa_2 - a_2 S(t) & -\tfrac{1}{2}(a_3 + b_2)S(t) & -\tfrac{1}{2}(a_4 + c_2)S(t) \\ \hline 0 & -\tfrac{1}{2}(a_3 + b_2)S(t) & -\epsilon - b_3 S(t) & -\tfrac{1}{2}(b_4 + c_3)S(t) \\ \hline 0 & -\tfrac{1}{2}(a_4 + c_2)S(t) & -\tfrac{1}{2}(b_4 + c_3)S(t) & -\epsilon - c_4 S(t) \\ \hline \end{array}$$

For asymptotic stability of the trivial solution of (30), the matrix $\underline{A}$ must be positive definite for all $t \geq 0$.

The _following algorithm_ [10] is similar to the above work. Here we consider the system

$$\dot{\underline{x}} = \underline{A}(t)\ \underline{x}\ , \tag{31}$$

where $\underline{A}(t)$ is real. Since $\underline{A}$ is a square matrix, there exists a transformation matrix $\underline{T}(t)$ such that the similarity transformation $(\underline{T}^{-1}\ \underline{A}\ \underline{T})$ takes $\underline{A}(t)$ into a Jordan canonical form; that is, the matrix $\underline{J}(t) = \underline{T}^{-1}\ \underline{A}\ \underline{T}$ consists of one or more Jordan blocks which are square arrays of the form

$$\begin{array}{|c|c|c|c|c|c|} \hline \lambda & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & \lambda & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & \lambda & 1 & \cdots & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \hline 0 & 0 & 0 & 0 & \cdots & \lambda \\ \hline \end{array}\ ,$$

where $\lambda$ is an eigenvalue of $\underline{A}$. These Jordan blocks are located along the principal diagonal of $\underline{J}(t)$ and all the elements not contained in the Jordan blocks are equal to zero.

We define the matrix $\underline{P}(t)$ by

$$\underline{P}(t) = \left[\underline{T}(t)\ \underline{T}^{*}(t)\right]^{-1}\ ,$$

where (*) denotes the conjugate transpose of $\underline{T}$. The Liapunov function to be used in the stability studies is defined as

$$V\ (\underline{x}, t)\ =\ \underline{x}^{*}\ \underline{P}\ (t)\ \underline{x}\ .$$

From the properties of the transformation matrix $\underline{T}$, it can be shown that V is bounded and positive definite. The time derivative of V is given by

$$\dot{V}\ =\ -\ \underline{x}^{*}\ \underline{Q}(t)\ \underline{x}\ ,$$

where $\underline{Q}(t)\ =\ \left[\underline{T}^{-1}\right]^{*}\ \underline{M}(t)\ \underline{T}^{-1}$ and $\underline{M}(t)\ =\ -\ \left[\underline{J}(t)\ +\ \underline{J}^{*}(t)\right]\ +\ \left[(\underline{T}^{-1}\ \dot{\underline{T}})\ +(\underline{T}^{-1}\ \dot{\underline{T}})^{*}\right]$.

If $\underline{M}(t)$ is positive semidefinite, the system is <u>stable</u>. If $\underline{M}(t)$ is positive definite, the system is <u>asymptotically stable</u>.

## Example

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a(t) & a(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

The matrices $\underline{J}(t)$ and $\underline{T}(t)$ are given by:

$$\underline{J}(t)\ =\ \begin{bmatrix} 0 & 0 \\ 0 & a(t) \end{bmatrix} \quad ,\quad \underline{T}(t)\ =\ \begin{bmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1 \end{bmatrix}$$

The resulting $\underline{M}$ matrix is

$$\underline{M}(t)\ =\ -\ \left[\underline{J}\ +\ \underline{J}^{*}\right]\ =\ \begin{bmatrix} 0 & 0 \\ 0 & -2a(t) \end{bmatrix} \ ;$$

thus, the system is <u>stable</u> when $a(t)\ \leq\ 0$ for $t\ \geq\ 0$.

**NOTE** There are many cases where $\underline{M}$ reduces to the form $\underline{M}\ =\ -\ \left[\underline{J}\ +\ \underline{J}^{*}\right]$. This form occurs when $\underline{A}(t)$ is time invariant, $\underline{A}(t)$ is symmetrical for all $t\ \geq\ $ to, or when $\underline{A}\ =\ \underline{B}\ \underline{C}$, where $\underline{B}$ is a time varying symmetrical matrix and $\underline{C}$ is a time invariant positive definite matrix, or vice - versa.

We give two variations of this method, [10]. Let V($\underline{x}$, t) be the Hermitian form

$$V(\underline{x}, t) = e^{\eta(t)} \underline{x}^* \underline{P}(t) \underline{x},$$

$$\dot{V}(\underline{x}, t) = - \underline{x}^* \underline{Q}(t) \underline{x},$$

where $\underline{P}$ is given above and $\eta(t)$ is such that $\underline{Q}$ is "just" positive semidefinite. The $\underline{Q}$ - matrix is given by

$$Q(t) = \left[ \underline{T}^{-1} \right]^* \left\{ - (\underline{J} + \underline{J}^*) + \underline{T}^{-1} \dot{\underline{T}} + (\underline{T}^{-1} \dot{\underline{T}})^* - \dot{\eta}(t) \underline{I} \right\} \underline{T}^{-1}.$$

We choose $\eta(t)$ as the instantaneous minimum of the solutions, $\lambda$, of the equation

$$\text{Det} \left\{ - (\underline{J} + \underline{J}^*) + \underline{T}^{-1} \dot{\underline{T}} + (\underline{T}^{-1} \dot{\underline{T}})^* - \lambda \underline{I} \right\} = 0.$$

thus, the system (31) is stable if

$$\lim_{T \longrightarrow \infty} \left[ \eta(T) - \eta(to) \right] > 0.$$

The other variation, [10], is as follows:

as before we define $\underline{T}(t)$ by the equation $\underline{J} = \underline{T}^{-1} \underline{A} \underline{T}$ and we let V($\underline{x}$, t) = $\underline{x}^* \underline{P}(t) \underline{x}$. The time derivative of V is taken to be $\dot{V}(\underline{x}, t) = - \underline{x}^* \underline{Q}(t) \underline{x}$; but instead of defining $\underline{P}(t)$ by $\underline{T}^* \underline{P} \underline{T} = \underline{I}$, we let $\underline{T}^* (\underline{Q} + \dot{\underline{P}}) \underline{T} = \underline{I}$ and define the matrix $\underline{D}(t)$ as $\underline{D} = \underline{T}^* \underline{P} \underline{T}$. Now if Re($\underline{J}$) is a non-singular matrix, $\underline{D}(t)$ can be computed from the formula $\underline{J}^* \underline{D} + \underline{D} \underline{J} = - \underline{I}$ and $\underline{Q}$ is given by

$$Q(t) = (T^*)^{-1} \left\{ \underline{I} + \dot{\underline{D}} - \underline{D} \underline{T}^{-1} \dot{\underline{T}} - (\underline{T}^{-1} \dot{\underline{T}})^* \underline{D} \right\} \underline{T}^{-1}.$$

Then if it happens that $\underline{Q}$ is positive definite for all t $\geqslant$ to, the following criterion on $\underline{D}$ will determine the stability of the system:

(1) if $\underline{D}$ is positive definite for all t $\geqslant$ to and decrescent (i.e., $\underline{x}^* \underline{D} \underline{x} \longrightarrow 0$ uniformly in t as $\underline{x}^* \underline{x} \longrightarrow 0$), the system is asymptotically stable;

(2) if any eigenvalue of $\underline{D}$ is negative for all t $\geqslant$ to, the system is unstable.

The <u>following algorithm</u> is due to Szego, [12]. Consider the linear, nonautonomous system

$$\dot{\underline{x}} = \underline{A}(t) \; \underline{x} \; , \tag{32}$$

and introduce the following quadratic form

$$V \; (\underline{x}, \; t) = \underline{x}_T \; \underline{c}(t) \; \underline{x} \; ,$$

where $\underline{c}(t)$ is the solution of the equation

$$\underline{C}(t) \; \underline{A}(t) \; + \left[ \underline{C}(t) \; \underline{A}(t) \right]_T = \underline{C}(t) \cdot s \cdot \underline{B}(t) - \mathcal{V}(t) \; \underline{I}. \tag{33}$$

The function $\mathcal{V}(t)$ is a positive, differentiable scalar function; $\underline{B}$ is a symmetric matrix to be determined; ($\underline{c} \cdot s \cdot \underline{B}$) denotes the Schur's product of $\underline{C}$ and $\underline{B}$ (i)e., the matrix formed by multiplying the corresponding elements of $\underline{C}$ and $\underline{B}$); and the matrix $\underline{C}$ is symmetric. The solution of (33) has the form

$$\underline{C}(t) = \mathcal{V}(t) \; \underline{G} \; (a_{ij}, \; b_{ij}) \; ,$$

where $\underline{A} = \left[ a_{ij} \right]$ and $\underline{B} = \left[ b_{ij} \right]$. We shall denote $\underline{G} \; (a_{ij}, \; b_{ij})$ by $\underline{G}(t)$. When computing $\underline{G}(t)$ we fix some elements of $\underline{B}(t)$ we fix some elements of $\underline{B}(t)$ to assure that $g_{ii} < 0$ (i = 1, ..., n) for all t $\geqslant$ to. The other elements of $\underline{B}$ are arbitrary and one then can compute $\underline{c}$ from (33), where $\mathcal{V}$ is to be determined later.

$\dot{V}(\underline{x}, t)$ will be negative definite if the matrix $\underline{D}(t)$, given by

$$\underline{D}(t) = \underline{G}(t) \cdot s \cdot \underline{B}(t) + \dot{\underline{G}}(t) + \phi(t) \; \underline{G}(t),$$

where $\phi(t) = \dot{\mathcal{V}}(t) / \mathcal{V}(t)$, is positive semi-definite. One chooses the value of $\phi(t)$ such that $\underline{D}$ is positive semidefinite and then computes $\mathcal{V}(t)$. Thus, $\dot{V}$ is negative definite by the choice of $\phi$; then the matrix $\underline{G}(t)$ is examined to determine if $V(\underline{x}, t)$ is positive definite. In summary, the <u>sufficient conditions for asymptotic stability</u> of (32) are:

$$\left| G_{2K}(t) \right| > 0, \quad K = 1, \; ..., \; \frac{n}{2} \; \text{for t} \; \geqslant \; \text{to, and even n}$$

$$\left| G_{2K+1}(t) \right| > 0, \quad K = 1, \; ..., \; \frac{n-1}{2} \; \text{for t} \; \geqslant \; \text{to, and odd n;} \tag{34}$$

and if the order of $\underline{G}$ is even, $\left| \underline{G} \right| > \delta > 0$ for all t $\geqslant$ to, and if the order

of $\underline{G}$ is odd, $\left| \underline{G} \right| < - \delta < 0$ for all $t > $ to; and $- \infty < \int_{to}^{t} \phi(t) \, dt$

for all $t \geqslant$ to. The $\left| \underline{G}_i(t) \right|$ $(i = 1, \ldots, n)$ are the principal minors of the

determinant $\left| \underline{G} \right|$.

## Example $\begin{bmatrix} 12 \end{bmatrix}$

Consider the system:

$$\dot{x}_1 = x_2 \, ,$$

$$\dot{x}_2 = P_1(t)x_1 + P_2 x_2 \, ;$$

where $P_1(t)$ is a differentiable, bounded, always negative, and decreasing function

of t, and $P_2$ is a negative constant. The form of $V(\underline{x}, t)$ is:

$$V(\underline{x}, t) = c_{11}(t) \, x_1^2 + 2c_{12}(t) \, x_1 x_2 + c_{22}(t) \, x_2^2 \, .$$

From (33), we compute $\underline{G}$. For simplicity, let $b_{12} = b_{22} = 0$, then

$$g_{11}(t) = \frac{2}{\delta(t)} \left\{ P_1(t) - P_1^2(t) - P_2^2 \right\}$$

$$g_{12}(t) = \frac{1}{\delta(t)} \left\{ 2P_2 - b_{11}(t) \, P_1(t) \right\}$$

$$g_{22}(t) = \frac{1}{\delta(t)} \left\{ P_2 b_{11}(t) + 2 \, P_1(t) - 2 \right\}$$

where $\delta(t) = 2 \left\{ 2P_1 P_2 + b_{11} \left[ P_2^2 - P_1 \right] \right\}$ .

Choose $b_{11}$ such that V is positive definite and then examine the restrictions which

must be placed on the system to make $\dot{V}$ negative definite. To satisfy (34), let

$b_{11} = 2P_1^2$ . The other conditions for V to be positive definite reduce to

$$g_{11} \, g_{22} - g_{12}^2 > \eta \quad \text{for t} \geqslant \text{to,}$$

where $\eta$ is a positive constant. This inequality yields the following condition on the

$P_i$'s:

$$2p_1^2 - p_1^3 - p_1 p_2^2 + 3p_1^3 p_2 - p_1^4 p_2 - p_1^2 p_2^3 +$$

$$- p_1 - p_1^6 > \eta \text{ for } t \geqslant t_0.$$

(35)

The remainder of the computational procedure will now be outlined.

The $\underline{D}(t)$ matrix can be expressed in terms of the elements of $\underline{G}$, $\underline{B}$ and the unknown function $\phi$. The semidefiniteness of $\underline{D}$ and equation (35) impose restrictions on the $\phi$ - function. From these restrictions on $\phi$, the $\psi$ - function can be determined. Finally, the $\underline{C}$ matrix can be determined from $\psi$ and G. Thus, from the $V(\underline{x}, t)$ function, formed from matrix $\underline{C}$, the conditions for the stability of $\underline{x} = \underline{0}$ can be decided.

## Example [12]

Szegö found that for systems of the form studied in the above example, a simplified procedure can be followed. Consider the second order system given by:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = a_1(t) x_1 + a_2(t) x_2.$$

(36)

Let $V(\underline{x}, t)$ have the form

$$V = x_1^2 + c_{22}(t) x_2^2.$$

Then, $\dot{V}$ becomes

$$\dot{V} = 2(1 + c_{22}a_1) x_1 x_2 + (2c_{22}a_2 + \dot{c}_{22})x_2^2.$$

For a semidefinite $\dot{V}$, let $1 + c_{22}a_1 = 0$, and $2c_{22} a_2 + \dot{c}_{22} < 0$. Thus, the resulting sufficient conditions for stability are

$$\left\{ - \frac{2a_2}{a_1} + \frac{\dot{a}_1}{a_1^2} \right\} < 0, \text{ for } t \geqslant t_0,$$

$$-\left\{ \frac{1}{a_1} \right\} > K_2(\text{constant}) > 0, \text{ for } t \geqslant t_0.$$

## Special Case of (36)

Consider the system

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \left\{\frac{1}{t} - a\right\} x_1 + \left\{\frac{b}{t(at - 1)} \sin(ct)\right\} x_2 \ . \tag{37}$$

The stability conditions are

$$2b \sin ct - 1 \leq 0 \ ,$$

$$\frac{t}{at - 1} > K_2 > 0 \ .$$

The conclusion is that the solution $x = 0$ of the system in (37) is _stable_ if

$b \leq 1/2$ and $ato > 1$ .

We now consider the _work of Kupsov_, reference [13], in which he considers the

following system:

$$
\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix}
=
\begin{vmatrix} a_{11}^{(t)} & a_{12}^{(t)} \\ a_{21}^{(t)} & a_{22}^{(t)} \end{vmatrix}
\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \ . \tag{38}
$$

## Theorem 30 [13]

"If there exists a constant M and a positive function $S(t)$ which has a

continuous derivative on $(0, \infty)$ such that

$$\int_0^t \left\{ \left[ \left(\frac{\dot{S}}{2S} + a_{11} - a_{22}\right)^2 + \frac{1}{S}(a_{21} + Sa_{12})^2 \right]^{\frac{1}{2}} - \frac{\dot{S}}{2S} + a_{11} + a_{22} \right\} dt \leq M \tag{39}$$

for all $t > 0$, then the trivial solution of (38) is _stable relative to $x_1$_.

Further, if

$$\int_0^t \left\{ \left[ \left(\frac{\dot{S}}{2S} + a_{11} - a_{22}\right)^2 + \frac{1}{S}(a_{21} + Sa_{12})^2 \right]^{\frac{1}{2}} + \frac{\dot{S}}{2S} + a_{11} + a_{22} \right\} dt \leq M,$$

then the trivial solution of (38) is <u>stable relative to x2.</u>"

## Notes Concerning Kupsov's Work

(1) The proof of Theorem 30 is based on Liapunov theory and the choice for the Liapunov function is $V = \underline{x}_T \, \underline{P}(t) \, \underline{x}$,

Where $\underline{P} = $

| $A(t)$ | $B(t)$ |
|--------|--------|
| $B(t)$ | $C(t)$ |

.

(2) The necessary and sufficient conditions for the trivial solution of (38) to be stable relative to $x_1$ and $x_2$ are that $C(t) / |\underline{P}|$ and $A(t) / |\underline{P}|$ be bounded, respectively. It can be shown that $C / |\underline{P}|$ is bounded above by

$$K \exp\left\{ \int_0^t \left[ \left\{ \left[ \frac{\dot{S}}{2S} + a_{11} - a_{22} \right]^2 + \frac{1}{S} (a_{21} + S a_{12})^2 \right\}^{1/2} - \frac{\dot{S}}{2S} + a_{11} + a_{22} \right] dt \right\},$$

where K is a specified constant. A similar bound exists for $A / |\underline{P}|$.

(3) If $a_{21}/a_{12} < 0$ and has a continuous derivative on $(0, \infty)$, we can choose $S = - a_{21}/a_{12}$ and thus simplify (39).

(4) In the special case where equation (38) reduces to

$$\ddot{x} + P(t) \, \dot{x} + q(t) \, x = 0, \tag{40}$$

The conditions for <u>stability relative to x</u> of the trivial solution reduce to

$$q(t) > 0, \quad \int_0^\infty \left[ \left| p + \dot{q}/2q \right| - (p + \dot{q}/2q) \right] dt < \infty.$$

(5) Leonov, refer to [13], also studied (40) and obtained the following results for stability relative to x:

$$q(t) > 0, \quad p(t) + \dot{q}(t)/2q(t) \geqslant 0. \tag{41}$$

If (41) is satisfied and $\dot{q}(t)$ is bounded in $(0, \infty)$, then the trivial solution of

(40) is also <u>stable relative to x</u>.

(6)  If the Liapunov function of (40) is chosen to be $V = x^2 + 1/q(t)\,\dot{x}^2$,

then V will be positive definite when $q(t) > 0$. $\dot{V} = -\dot{q}/q^2 - 2\,p/q$. Thus,

the trivial solution of (40) will be stable if $q(t) > 0$ and

$$\frac{\dot{q}(t)}{q(t)^2} + 2\,\frac{P(t)}{q(t)} \geqslant 0.$$

## Work of Narendra and Goldwyn [14]

In this work the existence of "Common Liapunov Functions" for linear time-varying systems is discussed. The concept of the "Common Liapunov Function", C.L.F., was discussed in <u>Section 5</u> of this report and will not be repeated here. We will <u>first</u> summarize the topics which are discussed in reference [14]; and then we will give a few of the time-varying examples which are presented in [14]. The results are as follows.

(1)  For a <u>negative feedback system</u> with G(s) in the forward path and a gain

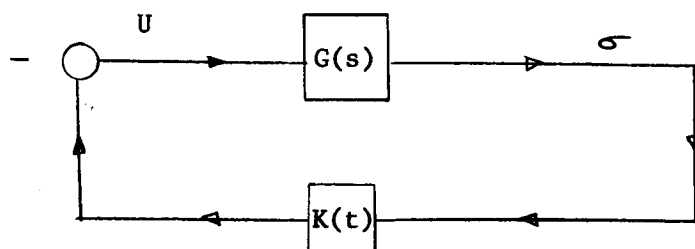K(t) in the feedback path $(0 \leq K(t) \leq \overline{K})$, see figure #1,

<u>Figure #1</u>

Time-Varying System

it is shown that a sufficient condition to ensure the existence of a C.L.F. and hence stability is that $\left[1/\overline{K} + G(s)\right]$ be a positive real function. A geometrical interpretation of the above condition yields a simple and effective method of determining the range of stability from the frequency response of the time-invariant part of the system.

(2)  For specific time-varying systems, Liapunov functions that are explicit functions

of time are found to increase the stability range of a parameter over that given

by the C.L.F. An analysis of the behavior of the Liapunov function V in the

V-$\dot{V}$ phase plane yields further insight into the problem of stability and leads

to the generation of Liapunov functions for an additional class of time-varying

systems.

(3) The problem of determining the entire range within which periodically varying

parameters may lie while assuring stability is intimately related to Floquet

theory. The generation of time-varying Liapunov functions in such cases requires

the solution of a matrix differential equations of the form

$$\dot{\underline{P}} + \underline{F}_T \underline{P} + \underline{P} \underline{F} = - \underline{Q} .$$

The origin of this matrix equation is as follows: consider the system

$$\dot{\underline{x}} = \underline{F}(t) \underline{x}$$

and choose as a candidate for a Liapunov function $V = \underline{x}_T \underline{P}(t) \underline{x}$. One then obtains

$$\dot{V} = - \underline{x}_T \underline{Q}(t) \underline{x} ,$$

where $\underline{Q}$ is chosen to be semi-definite. Thus, we obtain the above relationship

between $\underline{P}$, $\underline{F}$, and $\underline{Q}$.

**Example** [14]

Consider the system

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = - \left\{ 1 + \zeta(t) \right\} x_1 - 2 \zeta x_2,$$

$$\zeta < 1/\sqrt{2} .$$

A C.L.F. can be found for the system when $\left| \zeta(t) \right| \leq 2 \zeta \sqrt{1 - \zeta^2}$ . The C.L.F.
may be taken as

$$V = \underline{x}_T \begin{bmatrix} 1 & \zeta \\ \zeta & 1 \end{bmatrix} \underline{x} ,$$

and

$$\dot{V} = - \underline{x}_T \begin{bmatrix} 2\zeta(1 + \iota(t)) & 2\zeta^2 + \iota(t) \\ 2\zeta^2 + \iota(t) & 2\zeta \end{bmatrix} \underline{x} .$$

For $\left| \iota(t) \right| < 2 \zeta \sqrt{1 - \zeta^2}$, $\dot{V} < 0$ for any $\underline{x}$ and hence the null solution is asymptotically stable. (It can be shown that no non-trivial half-trajectories of the system lie on the set of points for which $\dot{V} \equiv 0$.)

## Example [14]

Consider the equation

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = - b(t)x_1 - ax_2.$$

Let $V(\underline{x}, t)$ be of the form

$$V = \underline{x}_T \begin{bmatrix} a^2 & a \\ a & 2 \end{bmatrix} \underline{x} + 2 B(t) x_1^2 .$$

(i) For the case when

$$B(t) = 1/t \int_0^t b(u) \, du,$$

we have

$$\dot{V} = - 2 \underline{x}_T \begin{bmatrix} ab - \dot{B} & b - B \\ b - B & a \end{bmatrix} \underline{x} .$$

For $b(t) > \dot{B}/a + \left[ (b - B)/a \right]^2 > 0$, the above system is asymptotically stable.

(ii) For the case when $B(t) = b(t)$, the condition for asymptotic stability is

$$b > \frac{\dot{b}}{a} > 0.$$

## Work of Rohrer [15] and [18]

In reference [18], Rohner considers the undriven, single loop, linear time-variable network in Figure 2. The terms in the
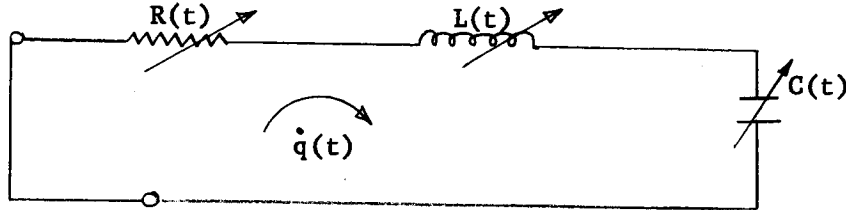


Figure 2

circuit are defined as:

$\dot{q}(t) \equiv$ unknown current

$R(t) \equiv$ time-varying resistance

$L(t) \equiv$ time-varying inductance

$C(t) \equiv$ time-varying capacitance.

Rohner uses the Hamiltonian formulation of analytical mechanics to determine the stability criteria of this time-varying electrical network. Upper and lower bounding functions for the network's stored energy are obtained which lead to sufficient conditions for network stability. The term stability as employed here means asymptotic stability in the large of the zero state of the network — where, if an unexcited network is given an arbitrary initial stored energy distribution, E(to), there is a net decrease in this energy over a given time interval, E(t) < E(to), t > to.

We now give a very brief outline of this work. The defining equation for the system is

$$\frac{d}{dt} \left\{ L(t) \, \dot{q} \right\} + R(t) \, \dot{q} + \frac{1}{C(t)} \, q = 0.$$

From the analytical mechanics approach, the Hamiltonian, $\mathcal{H}(t)$, is given by

$$\mathcal{H}(t) = \exp \left\{ \int_{to}^{t} \left\{ \frac{R(x)}{L(x)} \right\} dx \right\} E(t) ,$$

where E(t) is the stored energy in the network energy storage elements. It can be shown by considering certain properties of $\mathcal{H}$ that

$$E(t) \leq E(to) \quad \frac{C(to)}{C(t)}$$

when

$$- \frac{\dot{C}}{C} + \frac{\dot{L}}{L} + \frac{2R}{L} \geqslant 0.$$

Then $C(t) E(t)$ is a <u>Liapunov function</u>; and for positive R(t), the above inequality yields the Liapunov stability criterion

$$0 < 1 - \frac{1}{2} \left\{ \frac{\dot{C}L}{RC} - \frac{\dot{L}}{R} \right\}.$$

(Thus, we have presented only a brief outline of the work in [18] and is in no way a complete summary of Rohner's stability results.)

In reference [15] , Rohner considers a general R(t) - L(t) - C(t) linear, continuous time network. He studies the stored energy of the network, by considering upper and lower bounding functions, to provide sufficient conditions for stability and instability. In particular, if the stored energy decreases on the average, the network is stable; conversely, if the stored energy increases on the average, the network can be said to be unstable. This stability can be considered to be "asymptotic stability in the large." In the following paragraphs, we briefly outline Rohner's procedure. (This procedure is related to Liapunov theory and thus is presented in this report.)

The <u>energy analysis on the loop basis</u> starts with the second-order matrix equation:

$$\frac{d}{dt} \left\{ \underline{L}(t) \; \dot{\underline{q}} \right\} + \underline{R}(t) \; \dot{\underline{q}} + \underline{S}(t) \; \underline{q} = \underline{0}. \tag{42}$$

The terms in (42) are defined as follows:

$$\underline{L}(t) \equiv n \times n \quad \text{inductance matrix,}$$

$$\underline{S}(t) \equiv n \times n \quad \text{susceptance matrix,}$$

$$\underline{R}(t) \equiv n \times n \quad \text{resistance matrix,}$$

$$\dot{\underline{q}}(t) \equiv n \times 1 \quad \text{link-current matrix,}$$

where $\underline{L}$ and $\underline{S}$ are symmetrical and positive definite. The network stored energy is given as

$$E(t) = \frac{1}{2} \left\{ \dot{q}_T \, \underline{L} \, \dot{q} + q_T \underline{S} \, q \right\} , \tag{43}$$

where

$$\dot{E} = -\frac{1}{2} \left\{ \dot{q}_T \, \dot{\underline{L}} \, \dot{q} - q_T \dot{\underline{S}} \, q + \right. \tag{44}$$

$$\left. + \, \dot{q}_T \left[ \underline{R} + \underline{R}_T \right] \dot{q} \right\} .$$

We introduce an arbitrary time dependent function $f(t)$, and integrate $d(f\,E)/dt$ from $(to)$ to $(t)$ to obtain the following:

$$f(t)\, E(t) = f(to)\, E(to) + \frac{1}{2} \int_{to}^{t} \dot{q}_T \left\{ \dot{f} \, \underline{L} + \right.$$

$$\left. - f \left[ \dot{\underline{L}} + \underline{R} + \underline{R}_T \right] \right\} \dot{q} \; dt +$$

$$+ \frac{1}{2} \int_{to}^{t} q_T \left\{ \dot{f} \, \underline{S} + f \, \dot{\underline{S}} \right\} q \; dt. \tag{45}$$

The formulation in (45) leads to the following upper and lower bounds on the network stored energy if the congruent matrices $\underline{A}(t)$ and $\underline{B}(t)$ satisfy the following conditions:

$$\underline{A}_T \, \underline{L} \, \underline{A} = \underline{I} \quad \text{(the unit matrix)} ,$$

$$\underline{A}_T \left\{ \dot{\underline{L}} + \underline{R} + \underline{R}_T \right\} \underline{A} = \underline{\Lambda}_1(t) ,$$

where $\underline{\Lambda}_1$ is a diagonal matrix which has the n roots of

$$\det \left\{ \lambda_1(t) \, \underline{L} - \dot{\underline{L}} - \underline{R} - \underline{R}_T \right\} = 0 \tag{46}$$

as entries; and

$$\underline{B}_T \, \underline{S} \, \underline{B} = \underline{I} ,$$

$$\underline{B}_T \, \dot{\underline{S}} \, \underline{B} = -\underline{\Lambda}_2(t),$$

where $\underline{\Lambda}_2(t)$ is the diagonal matrix which has the n roots of

$$\det\left\{\lambda_2(t)\,\underline{S}\,+\,\dot{\underline{S}}\right\}\,=\,0 \tag{47}$$

as entries. The upper and lower bounds on $E(t)$, for the proper choices of f(t) in (45), are given by:

$$E(to)\,\exp\left\{-\int_{to}^{t}\lambda_l(x)\,dx\right\}\,\le\,E(t)\le E(to)\,\exp\left\{-\int_{to}^{t}\lambda_u(x)\,dx\right\},$$

where $\lambda_l$ is the maximum root of the equations (46) and (47), and $\lambda_u$ is the minimum root.

We now outline <u>the energy analysis on the node basis</u>. The set of network equations is given by

$$\frac{d}{dt}\left\{\underline{C}(t)\,\dot{\underline{\phi}}\right\}\,+\,\underline{G}(t)\,\dot{\underline{\phi}}\,+\,\underline{\Gamma}(t)\,\underline{\phi}\,=\,\underline{0}, \tag{48}$$

where

$$\dot{\underline{\phi}}\,\equiv\,\text{n-vector of tree - branch voltages,}$$

$$\underline{C}(t),\,\underline{\Gamma}(t)\,\equiv\,\text{positive definite, symmetrical matrices.}$$

The stored energy can be written as:

$$E(t)\,=\,\frac{1}{2}\left\{\dot{\underline{\phi}}_T\,\underline{C}\,\dot{\underline{\phi}}\,+\,\underline{\phi}_T\,\underline{\Gamma}\,\underline{\phi}\right\}, \tag{49}$$

where

$$\dot{E}\,=\,-\,\frac{1}{2}\left\{\dot{\underline{\phi}}_T\,\dot{\underline{C}}\,\dot{\underline{\phi}}\,-\,\underline{\phi}_T\,\dot{\underline{\Gamma}}\,\underline{\phi}\,+\,\dot{\underline{\phi}}_T\left[\underline{G}\,+\,\underline{G_T}\right]\dot{\underline{\phi}}\right\}.$$

Proceeding as before, we obtain the following bounds:

$$E(to)\,\exp\left\{-\int_{to}^{t}\lambda_l'(x)\,dx\right\}\,\le\,E(t)\,\le\,E(t_o)\,\exp\left\{-\int_{to}^{t}\lambda_u'(x)\,dx\right\},$$

where $\lambda_l'$ is the maximum root and $\lambda_u'$ is the minimum root of the equations

$$\det\left\{\lambda_4(t)\,\underline{\Gamma}\,+\,\dot{\underline{\Gamma}}\right\}\,=\,0\,,$$

and

$$\det\left\{\lambda_3(t)\ \underline{C}\ -\ \underline{\dot{C}}\ -\ \underline{G}\ -\ \underline{G_T}\right\}\ =\ 0.$$

From the above bounds on the stored energy, we can obtain some simple ground-state stability criteria. The upper bound of $E(t)$ is given by the larger of $\lambda_u(t),\ \lambda'_u(t)$ at each value of t, call the value $U(t)$; and the lower bound is given by the smaller of $\lambda_\ell(t)$ and $\lambda'_\ell(t)$ at each value of t, call the value $\ell(t)$. If one considers the net decrease in stored energy over $[t_o,\ t_1]$ as an indication of stability, then a sufficient condition for stability is

$$\int_{t_o}^{t_1} U(x)\ dx\ >\ 0. \qquad (50)$$

In a similar way, a sufficient condition for instability is

$$\int_{t_o}^{t_1} \ell(x)\ dx\ <\ 0. \qquad (51)$$

For asymptotic stability (50) becomes

$$\lim_{t \longrightarrow \infty} \int_{t_o}^{t} U(x)\ dx\ >\ 0. \qquad (52)$$

For a periodically variable network where all element (in the electrical system) values vary with the same period T, thus $U(t)$ and $\ell(t)$ have the same period T, and the sufficient condition for stability becomes

$$\int_{t_o}^{t_o + T} U(x)\ dx\ >\ 0; \qquad (53)$$

that for instability becomes

$$\int_{t_o}^{t_o + T} \ell(x)\ dx\ <\ 0. \qquad (54)$$

<u>Example</u>     [15]            <u>Mathieu's Equation</u>

The normalized form of Mathieu's Equation is

$$\ddot{q} + (S + \epsilon \cos t)\, q = 0.$$

In this equation $L(t) = 1$ , $C(t) = \dfrac{1}{S + \epsilon \cos t}$ ,

and $R(t) = 0$. Over the first cycle the energy is bounded by

$$E(t) \le \begin{cases} E(0), & 0 \le t \le \pi \\[2ex] E(0) \left\{ \dfrac{S + \epsilon \cos t}{S - \epsilon} \right\}, & \pi \le t \le 2\pi \ , \end{cases}$$

and

$$E(t) \ge \begin{cases} E(0)\ \dfrac{S + \epsilon \cos t}{S + \epsilon}, & 0 \le t \le \pi \\[3ex] E(0)\ \dfrac{S - \epsilon}{S + \epsilon}, & \pi \le t \le 2\pi. \end{cases}$$

These bounds give no information about stability of this network except the "almost trivial" case where $\epsilon = 0$.

But if some fixed resistance R placed in series with the above network is considered, stability can be guaranteed. This resistance can be found as a function of S and $\epsilon$ by ascertaining the minimum R which assures that $E(2\pi) = E(0)$. From (53), this R is that which makes

$$\int_0^{2\pi} \left\{ \min\left[ 2R,\ \frac{\epsilon \sin t}{S + \epsilon \cos t} \right] \right\}\, dt = 0,$$

and it is given by

$$2R = \max\left\{ \frac{\epsilon \sin t}{S + \epsilon \cos t} \right\}.$$

therefore,

$$R > \frac{\epsilon}{2\sqrt{S^2 - \epsilon^2}} \tag{55}$$

for stability. This is the same result which has been derived from Liapunov analysis by Hahn.

## Work of Bongiorno    [19]

In reference [19], Bongiorno considers an analytical technique for establishing the stability of linear, lumped - parameter systems with periodically - varying parameters, by means of analytically or experimentally determined frequency response data. The theory as given is not Liapunov theory; but in one example, Bongiorno's results are justified by Liapunov theory. (For this reason, we include his work in this section of the report.) The basic time - varying system is given in figure 3:
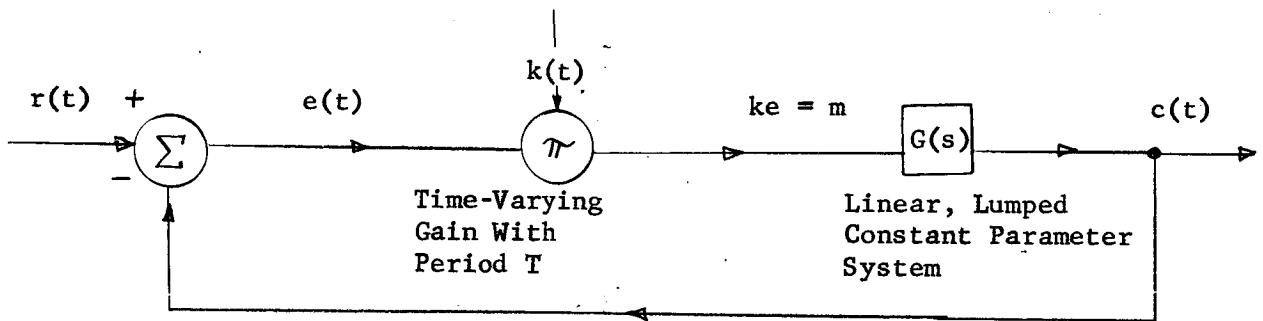


**Figure 3:**  Basic Time-Varying System

The application of the stability criterion yields easily obtainable bounds on the amplitudes of the periodically - varying parameters that are sufficient to insure system stability. The results which are obtained can in some cases be applied to systems with aperiodically - varying parameters, as was vigorously established for a certain case in reference [20].

The fundamental stability theorem of Borgiorno is:

## Theorem    [19]

"If in the system of figure 3, where $\left|k(t)\right| \leq K$ and $k(t)$ is periodic, the following conditions are satisfied:

(1)  for $K = 0$, the undriven system is asymptotically stable; and

(2)  for $0 < K \leq K_M$ , no steady-state solution of the form $e^{jwt} \phi(t)$ is possible,

then for all $0 \leq K \leq K_M$ the driven system is stable for all bounded inputs."

A sufficient condition for the satisfaction of condition 2 in the above theorem is

$$K \left| G(jw) \right|_{max} < 1 .$$
(56)

The derivation of this result is given in [19].

## Example    [19]

Consider the system

$$\ddot{x} + 2 \zeta w_o \dot{x} + w_o^2 \left\{ 1 - k(t) \right\} x = 0.$$
(57)

In this system, the expression for $G(s)$ is

$$G(s) = \frac{w_o^2}{s^2 + 2 \zeta w_o s + w_o^2} .$$

It can be shown that $\left| G(jw) \right|_{max}$ is given by

$$\left| G(jw) \right|_{max} = 1 \quad , \quad \zeta \geq 1/\sqrt{2} ,$$

$$= \frac{1}{2 \zeta \sqrt{1 - \zeta^2}} \quad , \quad \zeta \leq 1/\sqrt{2}.$$

Thus, from inequality (56), the conditions for asymptotic stability are

$$K < 1 \quad , \quad \zeta \geq 1/\sqrt{2},$$

$$< 2 \zeta \sqrt{1 - \zeta^2} \quad , \quad \zeta \leq 1/\sqrt{2} .$$
(58)

By means of Liapunov theory, it is "nearly possible" to reproduce the above stability conditions.  Liapunov's theory yields slightly more conservative results in this case.  We first consider two theorems before applying Liapunov's method to (57).

<u>Theorem</u>     [21]

"If the system is defined by

$$\ddot{x} + a\,\dot{x} + b(t)\,x = 0, \tag{59}$$

and if $0 < m \leq b(t) \leq M$, then the solutions of (59) are asymptotically stable for all (a) satisfying:

$$0 < \sqrt{M} - \sqrt{m} < a <$$

$$< \frac{M + 2\sqrt{m\,M} + 5\dot{m}}{\sqrt{M} - \sqrt{m}} \ . \ " \tag{60}$$

<u>Theorem</u>

"If   $0 < b(t) < a^2$, then the solutions of (59) are asymptotically stable."

<u>Proof</u>

Write (59) in the following form:

$$\dot{x}_o = x_1 \tag{61}$$

$$\dot{x}_1 = -ax_1 - b(t)\,x_o.$$

Choose the Liapunov function for (61) to be $V = x_o^2 + (2/a)\,x_o x_1 + (2/a2)\,x_1^2$.
The time derivative is

$$\dot{V} = -\frac{2}{a}\left\{ b\,x_o^2 + \frac{2b}{a}\,x_o x_1 + x_1^2 \right\}.$$

For asymptotic stability we require

$$b\,(1 - b/a^2) > 0, \tag{62}$$

or     $0 < b(t) < a^2.$                                        Q.E.D.

When $b(t) = W_o^2\left\{1 - k(t)\right\}$   and $a = 2\xi W_o$
in (59), (62) yields

$$0 < 1 - k(t) < 4\xi^2,$$

or

$$\left\{ \begin{array}{l} 1 + K < 4\xi^2 \\ 1 - K > 0 \end{array} \right\}. \tag{63}$$

The inequalities (63), obtained by Liapunov theory, are nearly the same as the inequalities given by Bongiorno's theory in (58).

Bongiorno in reference [19] also considers two other examples which will not be given here. One example considers more than one parameter varying with time; and the other considers a higher order time-varying system.

## REFERENCES

(1)  Brauer, F.: "Global Behavior of Solutions of Ordinary Differential Equations", _Jour. of Math. Anal. and Appl._, Vol. 2., pp 145-158, 1961.

(2)  Rosen, J. B.: "Stability and Bounds for Nonlinear Systems of Difference and Differential Equations", _Jour. of Math. Anal. and Appl._, Vol. 2, pp 370-393, 1961.

(3)  Rosenbrock, H. H.: "The Stability of Linear time-Varying Control Systems", _Jour. of Elec. and Control_, Vol. 2, pp 73-80, 1963.

(4)  Zubov, V.I.: _Mathematical Methods for the Study of Automatic Control Systems_, Macmillian Co., New York, 1963.

(5)  Zubov, V. I.: _Methods of A. M. Liapunov and their Applications_, A.E.C. - tr. - 4439, Office of tech. Services, Dept. of Comm., Wash. 25, D.C., 1957.

(6)  Kreidner, E.: "The Relation Between the Stability of Forced and Unforced Linear Systems", Appendix IV to Grumman Research Report- G.R.D. 121(1964).

(7)  Chethev, N. G.: "On Certain Questions Related to the Problem of the Stability of Unsteady Motion", _P.M.M._, Vol. 24, pp 5-22, 1960.

(8)  Persidskii, S. K.: "On the Second Method of Liapunov", _P.M.M._, Vol. 25, pp 20-28, 1961.

(9)  Roitenberg, T. N.: "A Method for the Construction of Liapunov Functions for Linear Systems with Variable Coefficients", _P.M.M._, Vol. 22, pp 230-236, 1958.

(10) Rohrer, R. A.: "Transformation techniques for the Application of Liapunov's Direct Method to Linear Systems", Unpublished Paper, University of Illinois.

(11) Bulgakov, B. V.: "On the Normal Coordinates", P.M.M., Vol. 10, p 280,(In Russian), 1946.

(12) Szegö, G.P.: "Method for Constructing Liapunov Functions for Linear Non-Stationary Systems", Jour. of Elec. and Control, Vol. 13, No. 1, p 69, 1962.

(13) Kupsov, N. P.: "On a Sufficient Condition for the Stability of the trivial Solution of a System of two Linear Differential Equations", P.M.M., Vol. 25, pp 1181-1186, 1961.

(14) Narendra, K. S. and Goldwyn, R. M.: "Generation of Quadratic type Liapunov Functions for Linear time - Varying Systems", Clearinghouse for Federal Scientific and technical Information, U. S. Dept. of Comm., A.D. 433982, 1963.

(15) Rohrer, R. A.: "Stability of Linear, time-Varying Networks --- Bounds on the Stored Energy," Unpubl. Paper, University of Illinois.

(16) Onuchie, N.: "On the Uniform Stability of a Perturbed Linear System", Jour. of Math. Anal. and Appl. Vol. 6, pp 457-464, 1963.

(17) Saaty, T. L. and Bram, J.: Nonlinear Mathematics, McGraw - Hill Book Co., New York, pp 250-252, 1964.

(18) Rohrer, R. A.: "Analytic Mechanics and the Analysis of Linear, time-Varying Networks", Jour. of the Franklin Inst., Vol. 278, No. 2, Aug., 1964.

(19) Bongiorno, J. J., Jr.: "An Extension of the Nyquist - Barkhausen Stability Criterion to Linear Lumped - Parameter Systems with time Varying Elements," I.E.E.E. trans. on Auto. Control, Apr. 1963, pp 166-170.

(20) Bongiorno, J. J., Jr.: "On the Stability and Convengence Properties of Linear time - Varying Systems," Microwave Res. Inst., Poly. tech. Inst. of Brooklyn, Brooklyn, N. Y., MRI - Rep. P.I.B. MRI-994-62, Mar. 1962.

(21)  Starzinskii, V. M.:  "A Survey of Works on the Conditions of Stability of the Trivial Solution of a System of Linear Differential Equations with Periodic Coefficients", Am. Math. Soc. Trans. Ser. 2., Vol. 1, p 212, 1955.

MISCELLANEOUS SECTION

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## MISCELLANEOUS SECTION

SUMMARY

In this section we have considered several different methods of obtaining Liapunov functions. Some of the work in this section could have been placed in other sections of the report, but because the time element this work has "landed in" the miscellaneous section. The "physical structure" of this section takes the following form:

(1) a compendium of examples, 32 in number, covering references $[1]$ to $[52]$ ; the "examples" may discuss the results of a single paper, some part of a paper, or the results from several papers;

(2) a subsection titled "Random Contributions to Stability Theory" which includes 36 items which are briefly outlined; the discussion covers references $[53]$ to $[149]$ , including contributions from some Italian mathematicians during the years 1951 - 1961;

(3) a subsection outlining the contributions to stochastic stability, references $[150]$ - $[181]$ ;

(4) a subsection outlining the contributions to partial differential equations, $[182]$ to $[196]$ ; to differential-difference equations, $[197]$ - $[226]$ ; to topological dynamics and dynamic systems, $[227]$ to $[239]$ .

## COMPENDIUM OF EXAMPLES

Example 1, [2]     Third Order Example

In reference [1] , <u>Pliss</u> proved the following theorem.

## Theorem

(H)     (i) If, the system is defined by

$$\dddot{x} + f(\ddot{x}) + \dot{x} + x = 0;$$

(ii)   $f(y)$ is continuous and differentiable for all $y$;

(iii)  $f(y)$ satisfies a Lipschitz condition for all $y$;

(iv)   $f(0) = 0$, and $\dfrac{df}{dy} > 1$ for all $y$;

(C)     then the equilibrium solution $x = \dot{x} = \ddot{x} = 0$ is stable.

<u>Ogurtsov</u>, in reference [2], considers a more general case in the following theorem.

## Theorem

(H)     (i)   If, the system is defined by
$$\dddot{x} + f(\ddot{x}) + b\dot{x} + a x = 0;$$

(ii)   $a > 0$, $b > 0$ and constant;

(iii)  $f(y)$ is differentiable and continuous for all $y$;

(iv)   $f(y)$ satisfies a Lipschitz condition for all $y$;

(v)    $f(0) = 0$, and $\dfrac{df}{dy} > a/_b$ for all $y$;

(C)    then the null solution is asymptotically stable for arbitrary initial perturbations.

Proof

The equivalent system is given by

$$\dot{x} = y$$

$$\dot{y} = z - f'(x)\, y$$

$$\dot{z} = -by - ax.$$

The candidate for a Liapunov function is

$$2V = 2a \int_0^x x\, f'(x)\, dx + 2axy + by^2 + z^2,$$

where V is positive definite by hypotheses (ii) and (v). The time derivative of V corresponding to the above system is given by

$$\dot{V} = -\left\{ bf'(x) - a \right\} y^2 < 0 \quad , \quad y \neq 0 ,$$

$$= 0 \text{ for } y = 0.$$

Since 2V can be written as

$$2V = J(x) + z^2 + \left\{ \frac{ax}{\sqrt{b}} + \sqrt{b}\, y \right\}^2 ,$$

$$J(x) = 2a \int_0^x x\, f'(x)\, dx - \frac{a^2}{b} x^2 ,$$

we must check $\lim_{|x| \to \infty} J(x)$. If $\lim_{|x| \to \infty} J(x) = \infty$ , then V is infinitely large and all the level curves, V = constant, are closed, thus giving asymptotic stability in the whole space.

If $\lim_{|x| \to \infty} J(x)$ converges, then among the level surfaces of V there will be open surfaces. But it can be shown that in general there is still stability. The author considered the region defined by

$$V(x, y, z) \leq L , \quad |x| \leq N, \quad L > 0, \, N > 0.$$

He showed that for all t > 0, all trajectories of the system are inside the bounded region. Thus, the theorem is proved.

<u>Example 2,</u>  [2]    <u>Fourth Order System</u>

The fourth order example of <u>Ogurtsov</u> is also stated in theorem-form.

<u>Theorem</u>

(H)   (i)   If the system is given by

$$\ddddot{x} + f(\dddot{x}) + c\ddot{x} + b\dot{x} + ax = 0;$$

(ii)     $f(y)$ is continuous and differentiable for all $y$;

(iii)    $f(y)$ satisfies a Lipschitz condition for all $y$;

(iv)     $f(0) = 0$; c, a and b are positive constants;

(v)      $\dfrac{df}{dy} > \dfrac{b}{c}$ ;  $bc\,\dfrac{df}{dy} - b^2 - a\left(\dfrac{df}{dy}\right)^2 > 0$ ,

for all $y$;

(C)   then the zero solution is asymptotically stable for arbitrary initial

perturbations.

<u>Proof</u>

Let us consider the equivalent system

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = u - f'(y)\,z$$

$$\dot{u} = -cz - by - ax.$$

The candidate for a Liapunov function is

$$2V = (b^2 + ac)\,x^2 + 2bcxy + (c^2 - 2a)\,y^2 + 4ax z + 2by z +$$

$$+ c z^2 + 2bxu + 2cyu + 2u^2 + 2b \int_{0}^{y} f'(y)\,y\,dy ,$$

$$= \begin{bmatrix} x, & y, & z, & u \end{bmatrix} \begin{bmatrix} b^2 + ac & bc & a & b \\ bc & c -2a + b^2/c & b & c \\ a & b & c & 0 \\ b & c & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} +$$

$$+ 2b \int_o^y \left\{ f'(y) - b/c \right\} y \, dy .$$

From hypotheses (iv) and (v), we see that V is positive definite and

$V \longrightarrow \infty$ as $\| \underline{x} \| \longrightarrow \infty$. The time derivative of V with respect to the system is given by:

$$\dot{V} = - abx^2 - 2af'(y) \, x \, z - cf'(y) \, z^2 + b \, z^2,$$

$$= - a \left\{ bx^2 + 2f'(y) \, x \, z + \frac{\{f'(y)\}^2 z^2}{b^2} \right\} - cf' z^2 + b z^2 + \frac{a}{b} (f')^2 z^2,$$

$$= - a \left\{ \sqrt{b} \, x + \frac{1}{\sqrt{b}} f'(y) z \right\}^2 - \frac{1}{b} \left\{ bcf'(y) - b^2 - af'^2(y) \right\} z^2.$$

Thus, by hypotheses (iv) and (v), $\dot{V} < 0$ if x and z are nonzero and $\dot{V} = 0$ if x = z = 0. Therefore, the zero solution is asymptotically stable for arbitrary initial disturbances.

## Example 3, [3]    Ezeilo's Fourth Order Examples

In reference [3], Ezeilo discusses two examples; these examples will be written in the form of theorems.

## Theorem 1

(H)  (i)    If the system is defined by

$$\ddddot{x} + f(\ddot{x}) \, \dddot{x} + \alpha_2 \ddot{x} + g(\dot{x}) + \alpha_4 x = 0; \qquad (1)$$

(ii)    $\alpha_2$ and $\alpha_4$ are positive constants,

(iii)   $f(y)$ , and $g'(y)$ are continuous for all $y$,

(iv)   $g(0) = 0$   and there exists   $\alpha_1 > 0$ and   $\alpha_3 > 0$

such that $\dfrac{g(y)}{y} \geq \alpha_3$   , $y \neq 0$, and

$f(z) \geq \alpha_1$   for all $z$ ;

(v)   there exists a positive constant $\Delta_0 > 0$ such that

$$\left\{\alpha_1 \alpha_2 - g'(y)\right\} \alpha_3 - \alpha_1 \alpha_4 \quad f(z) \geq \Delta_0$$

for all $y$ and $z$ ;

(vi)   $g'(y) - \dfrac{g(y)}{y} \leq \delta_1$   , for all $y \neq 0$   ,

where $\delta_1 < 2\alpha_4 \dfrac{\Delta_0}{\alpha_1 \alpha_3^2}$ ;

(vii)   $$\left\{\dfrac{1}{z} \int_0^z f(x)\,dx\right\} - f(z) \leq \delta_2$$

for all $z \neq 0$, where   $\delta_2 < 2\left\{\Delta_0 \Big/ \alpha_1^2 \alpha_3\right\}$ ;

(C)   then every $x(t)$, solution of (1), is such that as $t \longrightarrow \infty$ ,

$(x, \dot{x}, \ddot{x}, \dddot{x}) \longrightarrow (0, 0, 0, 0)$ .

## Special Cases of Theorem 1

(1)   When $f = \alpha_1 = $ constant, (vii) is trivially fulfilled.  This is a case which was discussed by Ezeilo in reference [5].

(2)   When $f = \alpha_1$   and   $g = \alpha_3 \dot{x}$ , $\alpha_1$   and $\alpha_3$ being constants, the equation (1) reduces to the linear problem considered in the Routh-Hurwitz analysis.

## Theorem 2

(H) (i)  If, the system is defined by

$$\dddot{x} + f(\ddot{x})\,\dddot{x} + \alpha_2 \ddot{x} + g(\dot{x}) + \alpha_4 x = p(t) ;$$

(ii) hypotheses (i) $\longrightarrow$ (vii) in Theorem 1 are valid;

(iii)
$$\int_0^t \left| p(\tau) \right| d\tau \leq A < \infty \quad \text{for all } t \geq 0 ;$$

(C) then for any finite $x_0$, $y_0$, $z_0$, $w_0$, there is a finite constant $D = D(x_0, y_0, z_0, w_0)$ such that the unique solution $x(t)$ which is determined by $(x(0), \dot{x}(0), \ddot{x}(0), \dddot{x}(0)) \equiv (x_0, y_0, z_0, w_0)$ satisfies $|x| \leq D$, $|\dot{x}| \leq D$, $|\ddot{x}| \leq D$, $|\dddot{x}| \leq D$ for all $t \geq 0$.

If the following discussion we will give the Liapunov function used in the proofs of the theorems, and we will outline some of the major points of the arguments. But, since the entire proofs are very long, we will not rewrite them.

The state variable notation or the equivalent first order systems, used by the author are as follows: for equation (1),

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w,$$

$$\dot{w} = -w f(z) - \alpha_2 z - g(y) - \alpha_4 x ,$$

and for equation (2),

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w,$$

$$\dot{w} = -w f(z) - \alpha_2 z - g(y) - \alpha_4 x + p(t).$$

The Liapunov function used for both (1) and (2) is

$$2V = \alpha_4 d_2 x^2 + (\alpha_2 d_2 - \alpha_4 d_1) y^2 +$$

$$+ 2 \int_0^y g(\eta) d\eta + (\alpha_2 d_1 - d_2^2) z^2 + 2 \int_0^z N f(N) dN +$$

$$+ d_1 w^2 + 2 \alpha_4 x y + 2 \alpha_4 d_1 x + 2 d_2 y \int_0^z f(N) \, dN$$

$$+ 2 d_1 z g(y) + 2 d_2 yw + 2 z w, \tag{3}$$

where $d_1 = \epsilon + 1/\alpha_1$, $d_2 = \epsilon + \alpha_4/\alpha_3$, and $\epsilon > 0$.

## Lemma 1

"The function V defined in (3) satisfies the following conditions:

(1) $V(o, o, o, o) = 0$;

(2) there exists positive constants $D_1$, $D_2$, $D_3$ and $D_4$ depending on $\epsilon$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, $\delta_1$, $\delta_2$, and $\Delta_o$ such that

$$V \geqslant D_1 x^2 + D_2 y^2 + D_3 z^2 + D_4 W^2,$$

for all x, y, z, w provided $0 < \epsilon \leqslant \epsilon_1$, where $\epsilon_1$ is a function of $\alpha_i$, $\delta_j$ and $\Delta_o$."

The <u>proof</u> of this Lemma depends on the following important inequalities:

$$d_1 - 1/f(z) \geqslant \epsilon \text{ for all } z,$$

$$d_2 - \frac{\alpha_4 y}{\delta(y)} \geqslant \epsilon \quad \text{for all } y \neq 0,$$

$$\alpha_2 - d_1 g'(y) - d_2 f(z) \geqslant \frac{\Delta_o}{\alpha_1 \alpha_3} - D_o \epsilon,$$

for all y and z, and where $D_o$ is a function of the $\alpha_i$'s.

## Lemma 2

"There exists constants $D_6 > 0$, $D_7 > 0$, $D_8 > 0$ depending only on $\epsilon$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ and $\Delta_o$ such that for solutions of (1) we have

$$\dot{V} \leq - (D_6 y^2 + D_7 z^2 + D_8 w^2)$$

provided $0 < \epsilon \leq \epsilon_2$, where $\epsilon_2$ depends on $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\Delta_0$."

The proof of Theorem 1 is such that if $\epsilon \equiv \text{Min} \ (\epsilon_1, \epsilon_2)$, then V is shown to be a Liapunov function and equation (1) is globally asymptotically stable.

The proof of Theorem 2 makes use of the same V as in Theorem 1, but the time derivative of V varies; that is

$$(\dot{V})_{\text{Thm.2}} = (\dot{V})_{\text{Thm.1}} + (d_2 y + z + d_1 w) p(t).$$

Because of hypothesis (iii) in Theorem 2, we have that for any t,

$(\dot{V})_{\text{Thm.2}} \leq 0$. Thus, Theorem 2 is proved.

## Example 4, [4]    Ezeilo's Nonautonomous System

In this example we will consider a nonautonomous system defined by:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t), \tag{1}$$

where $\dot{\underline{x}}$, $\underline{x}$ and $\underline{f}$ are n-vectors. We assume that $\underline{f}$ and the Jacobian matrix of $\underline{f}$ exist and are continuous. The norm, $\|\underline{x}\|$, of $\underline{x}$ is defined by

$\|\underline{x}\|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$. As in the previous examples we will state the main results and merely outline the procedure used in the proofs.

## Theorem 1

(H)  (i)    If $\underline{A} = (\alpha_{ij})$ is a real, constant, symmetric, positive definite matrix;

(ii)    $\underline{J} = \left[ \dfrac{\partial f_i}{\partial x_j} \right]$ is the Jacobian matrix of $\underline{f}(x, t)$;

(iii) $\underline{D} = \underline{A} \cdot \underline{J} = (d_{ij})$ where each characteristic root

of $1/2 \, (\underline{D} + \underline{D}_T)$ satisfies,

$\lambda_K \leq -\delta < 0$ uniformly in x and for $t \geq t_0$;

(iv) $C_1 \equiv C_1 \, (\delta, t_0, \underline{A}, \underline{x}_0) > 0$,

$C_2 \equiv C_2 \, (\delta, \underline{A}) > 0$

$\propto \equiv$ largest characteristic root of $\underline{A}$,

p is a constant such that $1 \leq p \leq 2$,

$\mu$ is a constant such that $0 \leq \mu \leq \delta$ ;

(C) then every solution $\underline{x}(t)$ of (1) satisfies

$$\left\| \underline{x}(t) \right\| \leq \left\{ e^{-\left(\frac{p\mu}{\propto}\right)t} \left[ C_1 + C_2 \int_{t_0}^{t} \left\| \underline{f}(o, \tau) \right\|^p e^{\frac{p\mu}{\propto}\tau} \, d\tau \right] \right\}^{1/p} \quad (2)$$

for all $t \geq t_0$.

## Special Cases

(1) If $\underline{f}$ satisfies one or the other of

$$\max_{t \geq t_0} \left\| \underline{f}(0, t) \right\| < \infty, \text{ or } \int_{t_0}^{\infty} \left\| \underline{f}(0, t) \right\|^p dt < \infty \quad (3)$$

then the conclusion of Theorem 1 says that every $\underline{x}(t)$ satisfies

$\left\| \underline{x} \right\| \leq C$, where C depends on $\underline{f}$ and $\underline{A}$, and C is a finite,

positive constant. That is $\left\| \underline{x} \right\|$ is __bounded__ for all

$t \geq t_0$.

(2) If $\underline{f}$ satisfies $\underline{f}(\underline{0}, t) \equiv \underline{0}$ for all $t \geq t_0$, then Theorem 1 implies

that every $\underline{x}(t)$ satisfies $\left\| \underline{x}(t) \right\| \longrightarrow 0$ as $t \longrightarrow \infty$ .

In Theorem 2, Ezeilo specifies the sufficient conditions for

$\left\| \underline{x}(t) \right\| \longrightarrow o$ as $t \longrightarrow \infty$, but with $\underline{f}(\underline{0}, t) \not\equiv \underline{0}$.

## Theorem 2

(H)   (i)   If for any b, $0 \leq b < \infty$, $\underline{f}(\underline{x}, t)$ satisfies

$$0 \leq \| \underline{f}(\underline{x}, t) \| \leq \phi(b) \qquad \text{for} \quad \| \underline{x} \| \leq b$$

and uniformly in $t \geq t_o$;

(ii)   $\phi(b)$ is a continuous function of b;

(iii)   $\displaystyle\int_{to}^{\infty} \| \underline{f}(0, t) \|^{p} \, dt < \infty$, $p \in [1, 2]$;

(iv)   conditions on $\underline{A}$ and $\underline{f}$ given in Theorem 1 are valid;

(C)   then every solution of (1) satisfies $\underline{x}(t) \longrightarrow \underline{0}$ as $t \longrightarrow \infty$.

## Special Case

Let $\underline{f} = \underline{F}(x) + \underline{e}(t)$ where $\underline{F}$ and $\underline{e}$ are continuous vector-valued functions. Hypotheses (i) is satisfied if $\| \underline{e} \|$ is finitely bounded for all sufficiently large t, and (iii) is satisfied if

$$\int_{to}^{\infty} \| \underline{F}(\underline{0}) + \underline{e}(t) \|^{p} \, dt < \infty \qquad .$$

The result in the following Lemma is required for the determination of $\dot{V}$ in the proof of Theorem 1.

## Lemma

(H)   (i)   If $\underline{g}(\underline{x}, t)$ is a continuous real n-vector, with a continuous Jacobian matrix $\underline{G}$;

(ii)   N is a finite constant, $-\infty < N < \infty$;

(iii)   the characteristic roots of $1/2(\underline{G} + \underline{G}_T)$ are all less than or equal to N, uniformly in $\underline{x}$ and for $t \geq t_o$;

(C)   then for any $\underline{x}$  and $\underline{h}$, we have

$$\left[\underline{g}(\underline{x} + \underline{h},\ t) - \underline{g}\ (\underline{x},\ t)\right]_T \underline{h} \ \leq\ N\ \|\underline{h}\|^2 .\tag{4}$$

In the <u>proof of Theorem 1</u>, we consider the following positive definite form:

$$V(t) = \underline{x}_T\ \underline{A}\ \underline{x}\ ,\tag{5}$$

where $\underline{x}(t)$ is a solution of (1).  Since $\underline{A}$ is symmetric and positive definite, we have

$$\propto \|\underline{x}\|^2 \geqslant V(t) \geqslant \propto' \|\underline{x}\|^2\ ,$$

where $\propto\ >\ 0$      and      $\propto'\ >\ 0$          are the greatest and least characteristic roots of $\underline{A}$.  From the above Lemma and the hypotheses of Theorem 1, $\dot V$ is bounded from above in the following fashion:

$$\dot V\ \leqslant\ -\ \frac{2\sigma}{\propto}\ V\ +\ c_3\ \|\underline{f}\ (\underline{0},\ t)\|\ V^{1/2}\ ,$$

where $C_3$ depends on $\underline{A}$ and is positive.  By considering $V$ and $\dot V$, we can derive equation (2).

The <u>outline</u> of the <u>proof</u> of Theorem 2 is as follows:  The V-function of (5) is used; show that $\int_{t_0}^{t} V(\tau)\ d\tau\ =\ 0(1)$          as $t \longrightarrow \infty$ ; since $V \geqslant 0$, then $V(t)$ must approach zero as $t \longrightarrow \infty$; since $\propto \|\underline{x}\|^2\ V(t) \geqslant \propto' \|\underline{x}\|^2$          , then $\underline{x} \longrightarrow \underline{0}$ as $t \longrightarrow \infty$.

<u>Example 5, [6]</u>

In this paper Ezeilo considers the equation

$$\dddot{x}\ +\ f(x,\ \dot x)\ \ddot x\ +\ g(\dot x)\ +\ h(x)\ =\ 0.$$

He proved asymptotic stability in the large for the trivial solution, $x \equiv 0$, by assuming:

(1)   the generalized Routh-Hurwitz conditions are satisfied;

(2) sufficient conditions are satisfied such that the candidate for the Liapunov function is radially unbounded in the phase plane. (Because we have not seen this paper [6] , the above information is all that we can report.)

Example 6, [7]    Bellman's Vector Lyapunov Functions

The following discussion is an outline of Bellman's paper

on Vector Lyapunov Functions.

Bellman states that the second method of Liapunov depends upon the fact that a function satisfying the scalar inequality

$$\frac{du}{dt} \leq Ku, \quad u(0) = c$$

is majorized by the solution of the equation

$$\frac{dv}{dt} = Kv, \quad v(0) = c.$$

Bellman says that in some cases it might be more convenient to use a vector Liapunov function rather than a scalar function. If it is, then a vector analogue of the above majorization relation should exist. It has been proved that this analogue does exist.

We first consider a lemma for nonnegative matrices. Let $\underline{A}$ be a constant matrix and $e^{\underline{A}t}$ be the corresponding matrix exponential. It is known that $e^{\underline{A}t}$ is the solution of the matrix equation

$$\frac{d\underline{X}}{dt} = \underline{A} \, \underline{X}, \quad \underline{X}(0) = \underline{I},$$

where $\underline{X}$ is a square matrix. For the elements of $e^{\underline{A}t}$ to be nonnegative it is necessary and sufficient that $a_{ij} \geq 0, i \neq j$ .

Lemma

If $a_{ij} \geq 0, i \neq j$ , then

$$\frac{d\underline{x}}{dt} \leq \underline{A} \, \underline{x}, \quad \underline{x}(0) = \underline{C}, \quad \underline{x} \text{ (N - vector)} ,$$

implies $\underline{x} \leq \underline{y}$ where

$$\frac{d\underline{y}}{dt} = \underline{A}\ \underline{y}\ ,\quad \underline{y}(0) = \underline{C}.$$

(Here $\underline{x} \leq \underline{y}$ means component-by-component majorization.)

As an <u>application of this lemma,</u> let the two scalar functions, u and v, of t satisfy the inequalities

$$0 \leq U \leq K_1,\quad 0 \leq v \leq K_2, \tag{1}$$

and the differential equations

$$\dot{u} = -\,a_{11}u\ +\ a_{12}v\ +\ b_1 u\,v,\quad u(0) = c_1,$$

$$\dot{v} = \quad a_{21}u\ -\ a_{22}v\ +\ b_2 u\,v,\quad v(0) = c_2, \tag{2}$$

where $a_{ij} \geq 0,\ b_1, b_2 \geq 0,\ c_1, c_2 \geq 0$ . From Poincare-Liapunov Theory we have <u>local asymptotic stability</u> if the characteristic roots of

$$\underline{A} = \begin{array}{|c|c|} \hline -\,a_{11} & a_{12} \\ \hline a_{21} & -a_{22} \\ \hline \end{array} \tag{3}$$

have negative real parts and $C_1$ and $C_2$ are sufficiently small. Using the above Lemma, we can obtain a <u>nonlocal result</u>. From (1) and (2) we have

$$\dot{u} \leq -\ a_{11}u\ +\ a_{12}v\ +\ b_1 K_2 u\ ,$$

$$\dot{v} \leq \quad a_{21}u\ -\ a_{22}v\ +\ b_2 K_1 v\ ; \tag{4}$$

and the solutions of (4) are majorized by the solutions of

$$\dot{w} = -\,a_{11}w\ +\ a_{12}z\ +\ b_1 K_2 w\ ,\quad w(0) = C_1\ ,$$

$$\dot{z} = \quad a_{12}w\ -a_{22}z\ +\ b_2 K_1 z\ ,\quad z(0) = C_2; \tag{5}$$

that is, $0 \leq u \leq w$ and $0 \leq v \leq z$ . The solutions of (5) approach zero as $t \longrightarrow \infty$ if

$$\underline{B} = \begin{array}{|c|c|} \hline -a_{11}\ +\ b_1 K_2 & a_{12} \\ \hline a_{21} & -a_{22}\ +\ b_2 K_1 \\ \hline \end{array}$$

is a stable matrix. Therefore the Lemma implies that the solutions of (2) approach zero as $t \longrightarrow \infty$, as long as (1) is satisfied. This procedure can be generalized to higher order systems.

We now consider the <u>application of the Lemma to generating Liapunov functions</u>. Consider the system:

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{B}\ \underline{y} + \underline{g}\ (\underline{x},\ \underline{y})\ ,\ \underline{x}(0) = \underline{a}\ ,$$

$$\dot{\underline{y}} = \underline{C}\ \underline{x} + \underline{D}\ \underline{y} + \underline{h}\ (\underline{x},\ \underline{y})\ ,\ \underline{y}(0) = \underline{b}\ , \tag{6}$$

where $\underline{x}$ and $\underline{y}$ are m- and n-vectors, respectively and matrices $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ are constant and have appropriate dimensions. We now form two Liapunov functions

$$u = \underline{x}_T\ \underline{R}\ \underline{x}\ ,\quad v = \underline{y}_T\ \underline{S}\ \underline{y}\ , \tag{7}$$

where $\underline{R}$ and $\underline{S}$ are positive definite matrices. We assume that the bounds on $\underline{x}$ and $\underline{y}$ are known; thus, the constants corresponding to $\underline{g}$ and $\underline{h}$ in (6) and analogous to the $K_1$ and $K_2$ in (5) can be determined. Therefore, a majorized linear system corresponding to (6) can be obtained. Forming the vector Liapunov function, $[u, v]$, and making use of system (6) and its corresponding majorized system, we can form relationships for $[u, v]$ which are similar to (4) and (5). From the "majorized" system for the vector Liapunov function, $[u, v]$, a set of sufficient conditions for the stability of the null solution of (6) can be obtained.

<u>Example 7, [8]</u>   <u>Leighton's Second-Order Equation</u>

In this example we consider the equation

$$\ddot{x} = r\ (x,\ \dot{x}) \tag{1}$$

where $r(x, y)$ is of class $C^1$, continuous first partial derivatives, in a neighborhood R of (0, 0) and where $r(0, 0) = 0$. Equation (1) will be called <u>regular in R</u> if $\frac{\partial r(0,0)}{\partial y} \neq 0$. Associated with (1) is the system

$$\dot{x} = f(x,\ y) \tag{2}$$

$$\dot{y} = \frac{r\ [x,f(x,y)] - f(x,y)\ \frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}}\ \ )$$

where $f(x, y)$ is of class $C^1$ in R, $f(0, 0) = 0$, $\dfrac{\partial f(0,0)}{\partial y} \neq 0$ .

The critical points, or equilibrium solutions, of (2) are the solutions of the equations

$$f(x, y) = 0, \quad r(x, 0) = 0. \tag{3}$$

From (3) we see that the abscissas of the critical points of (2) are invariant under the various choices of f while the ordinates are not. We note that in most cases $f(x, y)$ can be taken as y, and thus system (2) becomes

$$\dot{x} = y$$
$$\dot{y} = r(x, y). \tag{4}$$

Further we suppose that $r(x, y) = 0$ only intersects the x - axis, in R, at the origin. Therefore, (0, 0) is an isolated critical point of (4) whose stability we wish to study.

In the following lemmas and theorems, an "LCL" function is a function, V, of x and y which determines the stability or in-stability of an isolated critical point of (4) by the theorems of Liapunov, Chetaev and LaSalle, as given in [9].

**Lemma**

"If (1) is regular, the function V defined by

$$2V = y^2 - 2 \int_0^x r(x, 0)\, dx \tag{5}$$

is an LCL function for the system (4) in the neighborhood of (0, 0)."

(The time derivative of (5) with reference to (4) is given by the formula:

$$\dot{V} = y \left\{ r(x, y) - r(x, 0) \right\} \Big).$$

The _proof_ of this Lemma will _not_ be repeated here. The summary of Leighton's discussion of the regular system (4) is stated in Theorem 1.

## Theorem 1

"If (1) is regular, and if (0, 0) is an isolated critical point of (4), this critical point is asymptotically stable if $\dfrac{\partial r(0,0)}{\partial y} < 0$ and if $xr(x, 0) < 0$, $x \neq 0$ . In all other regular cases, (0, 0) is unstable."

## Note

LCL functions for system (2) can be written in the form

$$2V = f^2(x, y) - 2 \int_0^x r(x, 0)\, dx. \tag{6}$$

The time derivative of V along the trajectories of (2) is given by

$$\dot{V} = f(x, y) \left\{ r\left[x, f(x, y)\right] - r(x, 0) \right\}. \tag{7}$$

It is possible that the V in (6) may be more useful and tractable than that provided by $f(x, y) = y$, but to find that optimum $f(x, y)$ is a difficult task.


## Special Cases

(1) In **Van der Pol's equation**

$$\ddot{x} = \epsilon(1 - x^2)\dot{x} - x, \quad \epsilon > 0, \tag{8}$$

we observe that

$$r(x, \dot{x}) = \epsilon(1 - x^2)\dot{x} - x,$$

and

$$\frac{\partial r}{\partial \dot{x}} = \epsilon(1 - x^2), \quad r(x, 0) = -x.$$

Thus, an equivalent system is

$$\dot{x} = y$$
$$\dot{y} = \epsilon(1 - x^2)y - x, \tag{9}$$

where $(0, 0)$ is an isolated critical point. Since $\dfrac{\partial r(0, 0)}{\partial y} = \epsilon > 0$ ,

then, by Theorem 1, (9) is <u>unstable at $(0, 0)$</u>.

If we now consider $\epsilon < 0$ ; and if we let $f(x, y) = y - \epsilon(x^3/3 - x)$,

then system (8) becomes

$$\dot{x} = y - \epsilon(x^3/3 - x)$$

$$\dot{y} = -x. \tag{10}$$

Leighton found that a <u>region of asymptotic stability</u> for system (10) is the

interior of $x^2 + y^2 = 3$; that is, the region is defined by

$$x^2 + \left\{\dot{x} + \epsilon(x^3/3 - x)\right\}^2 \leq 3 \tag{11}$$

where $x$ and $\dot{x}$ are taken as independent variables and $\epsilon < 0$ is a parameter

of the system of regions in (11).

(2) <u>Lienard's equation</u> is given as

$$\ddot{x} = -f(x)\dot{x} - g(x), \tag{12}$$

where

$$r(x, \dot{x}) = -f(x)\dot{x} - g(x),$$

$$r(x, 0) = -g(x),$$

$$\dfrac{\partial r(x, \dot{x})}{\partial \dot{x}} = -f(x).$$

An equivalent system is

$$\dot{x} = y,$$

$$\dot{y} = -y f(x) - g(x). \tag{13}$$

If we assume that $f$ and $g$ belong to $C^1$, $g(0) = 0$, $f(x) > 0$, and that

$xg(x) > 0$ for $x \neq 0$, then the origin is asymptotically stable by Theorem 1.

We are, of course, assuming $(0, 0)$ is the only critical point of (13).

The case $\dfrac{\partial r(0, 0)}{\partial y} = 0$ is considered in the following theorem,

theorem 2.

### Theorem 2

"Let $\ddot{x} = r(x, \dot{x})$ be such that $r$ is of class $C^1$ in some neighborhood $N$ of $(0, 0)$. Let $N^1$ be the neighborhood $N$ with $(0, 0)$ deleted, and suppose $(0, 0)$ is an isolated critical point of the system

$$\dot{x} = y$$

$$\dot{y} = r(x, y).$$

If $\dfrac{\partial r}{\partial y} > 0$ in $N^1$, the origin is unstable. If $\dfrac{\partial r}{\partial y} \le 0$ in $N^1$, the origin is stable if $xr(x, 0) < 0$ for all $x$ in $N^1$. This stability is asymptotic if $\dfrac{\partial r}{\partial y} < 0$ in $N^1$. If $\dfrac{\partial r}{\partial y}$ takes on both positive and negative values in every neighborhood of $(0, 0)$, then $(0, 0)$ may be either stable or unstable."

### Examples of the Last Conclusion in Theorem 2

Let us consider $\ddot{x} = x\dot{x} + x$. The corresponding first order system is

$$\dot{x} = y$$

$$\dot{y} = xy + x, \tag{14}$$

where $r(x, y) = xy + x$ and $\dfrac{\partial r}{\partial y} = x$. Since the linearized system corresponding to (14) has a positive characteristic root, system (14) is unstable at $(0, 0)$.

The system $\ddot{x} = x\dot{x} - x$, whose equivalent system is

$$\dot{x} = y$$

$$\dot{y} = xy - x \tag{15}$$

has a $\dfrac{\partial r}{\partial y} = -x$. The LCL function, (5), is $2V = y^2 + x^2$, where $V = xy^2$. Thus, we look for another $V$. Leighton's candidate for $V$ is given by

$$2V = x^2 - 2\left\{y + \log(1 - y)\right\} > 0$$

for sufficiently small y's. The time derivative of V is $\dot{V} \equiv 0$. Thus, (0, 0) is stable. But in both of the above examples, $\dfrac{\partial r}{\partial y}$ took on both positive and negative values in the neighborhood of (0, 0).

## Regions of Asymptotic Stability

Let us define N as the neighborhood of (0, 0) such that $\dfrac{\partial r(x, y)}{\partial y} < 0$ and $xr(x, 0) < 0$ for (x, y) in N. Next let $K_1$ be defined by

$$K_1 = \underset{\substack{x < 0 \\ x \in N}}{\text{L.U.b.}} \left\{ -2 \int_0^x r(x, 0) \, dx \right\}$$

and define $K_2$ and $K_0$ as

$$K_2 = \underset{\substack{x > 0 \\ x \in N}}{\text{L.U.b.}} \left\{ -2 \int_0^x r(x, 0) \, dx \right\} \qquad , \text{ and } K_0 = \min(K_1, K_2).$$

Then the region defined by

$$y - 2 \int_0^x r(x, 0) \, dx < K_0 \tag{16}$$

is a region of asymptotic stability of (0, 0).

## Example

Consider $\ddot{x} + a\dot{x} + 2bx + 3x^2 = 0$, a, b > 0. In this case, $r(x, y) = -ay - 2bx - 3x^2$. The set N is all points (x, y) for which $x > -\dfrac{2b}{3}$; and $K_0 = 8 \, b^3/27$ . Thus, a region of asymptotic stability is defined by

$$y^2 + 2bx^2 + 2x^3 < \frac{8 \, b^3}{27} \, ,$$

where $y = \dot{x}$.

<u>Example 8, [8]</u>    <u>Leighton's System of Two First-Order Equations</u>

Consider the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y), \tag{1}$$

where f is of class $C^2$ and g is of class $C^1$ in some neighborhood N of (0, 0).

We also assume that f(0, 0) = g(0, 0) = 0 and that the Jacobian of f and g

is nonzero at (0, 0). This insures that (0, 0) is an isolated critical point.

Further, assume that not both $\dfrac{\partial f(0,0)}{\partial x}$ and $\dfrac{\partial f(0,0)}{\partial y}$ are zero;

say, $\dfrac{\partial f(0,0)}{\partial y} \neq 0$. In order to determine an LCL function of (1) we

must determine $r(x, \dot{x})$, which is consistent with the above conditions placed

on f and g. First solve for y in the equation $\dot{x} = f(x, y)$; that is, $y = h(x, \dot{x})$.

Then, the resultant r is

$$r(x, \dot{x}) = \frac{g[x, h(x, \dot{x})] - \dot{x} \dfrac{\partial h(x, \dot{x})}{\partial x}}{\dfrac{\partial h(x, \dot{x})}{\partial \dot{x}}}. \tag{2}$$

This r satisfies the hypotheses of <u>Theorem 1 in Example 7</u>. Thus, the LCL

function is given by

$$V(x, \dot{x}) = \dot{x}^2 - 2 \int_0^x r(x, 0)\, dx. \tag{3}$$

<u>Special Case</u>

Consider the system

$$\dot{x} = -2x + y$$
$$\dot{y} = -y\sqrt{1 + x^2} \; .$$

The Jacobian of f and g is

$$J(f, g) = -2 \neq 0 \text{ at } (0, 0)$$

and $\dfrac{\partial f}{\partial y} = 1$ at (0, 0); that is, (0, 0) is an isolated critical point. Furthermore,
we have

$$y = h(x, \dot{x}) = \dot{x} + 2x,$$

$$r(x, \dot{x}) = \dot{x} \, (-2- \sqrt{1 + x^2}) - 2x \sqrt{1 + x^2},$$

$$\frac{\partial r}{\partial \dot{x}} = -2- \sqrt{1 + x^2} \quad, \quad \frac{\partial r(0, 0)}{\partial \dot{x}} = -3 \, ,$$

$$r(x, 0) = -2x \sqrt{1 + x^2} \quad.$$

Therefore, an LCL function is, by Theorem 1 in Example 7,

$$V = (\dot{x})^2 + 4 \int_0^x x \sqrt{1 + x^2} \quad dx,$$

where

$$\dot{V} = -2(2x - y)^2 (2 + \sqrt{1 + x^2}) \, .$$

Thus, the origin is asymptotically stable.

## Example 9, [8]    Leighton's Third Order Example

We consider the differential equation

$$\dddot{x} + \gamma(x,\dot{x}) \, \ddot{x} + \emptyset(x,\dot{x}) = 0, \tag{1}$$

and the associated system

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = -z \, \gamma(x,y) - \emptyset(x,y) \, , \tag{2}$$

where $\emptyset(0, 0) = 0$, and $\gamma$ and $\emptyset$ are of class $C^1$ near $(0, 0)$. Furthermore, $(0, 0)$ is assumed to be an isolated critical point of (2).

To study the stability of $(0, 0)$, Leighton considers the following Liapunov function:

$$2V = z^2 + 2 \int_0^y \emptyset(x, y) \, dy + 2\alpha \left\{ yz + \int_0^x \emptyset(x, 0) \, dx + \right.$$

$$\left. + \int_0^y y \, \gamma(x, y) \, dy \right\} \, , \tag{3}$$

where $\alpha$ is a constant to be chosen later. The time derivative of V with respect to (2) is

$$\dot{V} = z^2 \left\{ \alpha - \psi(x, y) \right\} + y^2 \left\{ \frac{1}{y^2} \int_0^y \left[ \frac{\partial \Theta(x, y)}{\partial x} + \right. \right.$$
$$\left. \left. + \alpha \left\{ y \frac{\partial \psi(x, y)}{\partial x} - \frac{\partial \Theta(x, y)}{\partial y} \right\} \right] dy \right\}. \tag{4}$$

The following lemma is a well known result, given in [9], but we will repeat it in the context of the above problem.

## Lemma

"If there exist a constant $\alpha$ such that V, in (3), is positive definite in a neighborhood of (0, 0, 0) while $\dot{V}$, in (4), is negative semi-definite in x, y, z and such that either $\dot{V} \equiv 0$ or $\dot{V} \neq 0$ along every nontrivial solution of (2), then the origin is locally stable. If $\dot{V} \not\equiv 0$, the origin is locally asymptotically stable. Finally, if $V \longrightarrow \infty$ as $x^2 + y^2 + z^2 \longrightarrow \infty$ is satisfied, the origin is completely stable."

In discussing V and $\dot{V}$ in the light of the above lemma, Leighton talks about a condition PH. His condition PH is: "Condition PH is satisfied if in a neighborhood of the origin $\psi(x, y) \geqslant \alpha$, V is locally positive definite , $y^{-1} J(x, y) \leq 0$, where J is the integral in (4), and if $\dot{V}$ does not vanish along the nontrivial solutions of the system in this neighborhood of (0, 0, 0)." The results of Leighton's investigation of (2) are given in the following theorem.

## Theorem 1

"If $\psi$ and $\Theta$ are of class $C^1$ and $\frac{\partial^2 \psi}{\partial x^2}$ , $\frac{\partial^2 \Theta}{\partial x^2}$ are continuous in a neighborhood of the origin, if $\Theta(0, 0) = 0, \psi(0, 0) > \alpha$,

$\frac{\partial \Theta(0, 0)}{\partial x} > 0, \frac{\partial \Theta(0, 0)}{\partial y} > 0$ , if $\psi(x, y) \geqslant \psi(0, 0)$,

if <u>condition PH</u> holds, and if

$$\frac{\partial \Theta(x, y)}{\partial x} + \alpha \left\{ y \frac{\partial \gamma(x, y)}{\partial x} - \frac{\partial \Theta(x, y)}{\partial y} \right\} \leq 0 .$$

near (0, 0, 0), then the origin is an asymptotically stable critical point of

system (2)."

## Special Cases

(1)   When $\gamma = f(y)$ and $\Theta = ay + bx$, we have the example given in [9], p. 71 .

Equation (3) becomes

$$V = \frac{b}{a} x^2 + a y^2 + z^2 + \frac{2b}{a} yz + 2bxy + \frac{2b}{a} \int_0^y yf(y) \ dy,$$

which is the same as given in [9]. The conditions of stability as given by

Theorem 1 are a $> 0$, b $> 0$, $\alpha = \frac{b}{a}$ , $f(y) > \frac{b}{a}$ .

(2)   Consider the differential equation

$$\dddot{x} + 3 \ddot{x} + 2 \dot{x} + x^3 = 0,$$

whose corresponding state variable form is

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = -3z - 2y - x^3.$$

The characteristic roots of the linearized system, about (0, 0, 0), are

0, -1, -2.   Thus, the <u>local stability</u> of (0, 0, 0) can <u>not</u> be determined in

this manner.   Applying Leighton's theorem we have:

$$\gamma (x, y) = 3, \quad \Theta(x, y) = 2 y + x^3$$

$$2V = z^2 + 6y z + 11y^2 + 2x^3 y + \frac{3}{2} x^4 ,$$

where   $\alpha = \gamma (0, 0) = 3.$

The time derivative of $V$ is $\dot{V} = -3y^2 (2 - x^2)$. Thus, $V$ is positive definite for $x^2 < 3/2$ and $\dot{V}$ is negative semi-definite; therefore, the origin is locally asymptotically stable.

(3) If $\not{\gamma} = a$ and $\Theta = by + cx$, we have a linear system. Leighton's Theorem indicates asymptotic stability of the origin if $a > 0$, $b > 0$, $c > 0$ and $ab > c$ hold. When $a > 0$, $b > 0$, $c > 0$ and $ab < c$, and taking $\propto = c/6$, Leighton's Theorem indicates an unstable origin. This is consistent with linear theory.

<u>Example 10, [8]</u>    <u>Leighton's System of Second Order Equations</u>.

Consider the system of differential equations

$$\ddot{x}_i = r_i (x_1, \ldots, x_n ; \dot{x}_1, \ldots, \dot{x}_n)$$
$$= r_i (\underline{x}, \underline{\dot{x}}),$$

(1)

where $i = 1, 2, \ldots, n$, and $r_i (\underline{x}, \underline{\dot{x}})$ vanish at $\underline{x} = \underline{\dot{x}} = \underline{0}$ and are of Class $C^1$ in some $2n$-dimensional neighborhood $N$ of the origin. Let $N^1$ denote the neighborhood $N$ with the origin deleted. For convenience let

$$R_i(\underline{x}) = R_i (x_1, \ldots, x_n) = r_i (\underline{x}; \underline{0})$$

and denote by $N_n$ the set of points $(\underline{x} ; \underline{0})$ in $N$. We suppose that the Jacobian

$$J_1 = \frac{\partial (r_1, \ldots, r_n)}{\partial (x_1, \ldots x_n)} \neq 0$$

(2)

at the origin, and thus in the neighborhood $N$. Therefore, from (2) we have that the origin is an isolated critical point of the system

$$\dot{x}_i = y_i,$$
$$\dot{y}_i = r_i (\underline{x} ; \underline{y}),$$

(3)

where $i = 1, 2, \ldots, n$; and (3) is the system associated with (1).

Next, we introduce the line integral

$$I(\underline{x}) = \int_{\underline{0}}^{\underline{x}} R_i(\underline{x}) \, dx_i \qquad \text{(i summed, 1 to n),} \quad (4)$$

and we assume that

$$\frac{\partial r_i}{\partial x_j} = \frac{\partial r_j}{\partial x_i} \qquad (5)$$

in N. Thus, $I(\underline{x})$ is independent of path in $N_n$. The results of the stability analysis of system (3) are summarized in the following theorem.

Theorem

"If the function $I(\underline{x})$ is positive definite and if the Jacobian

$$J_2 = \frac{\partial(r_1, \ldots, r_n)}{\partial(x_1, \ldots, x_n)} \qquad (6)$$

evaluated at $\underline{\dot{x}} = \underline{0}$ is the determinant of a negative definite quadratic form, then the function

$$2V = \sum_{i=1}^{n} y_i \, y_i - 2 \, I(\underline{x}) \qquad (7)$$

is a Liapunov function for the system (3), and the origin is a stable critical point of this system. (In fact using LaSalle's results, [9], the origin is asymptotically stable.)"

Note

Computing $\dot{V}$ along the trajectories of system (3) results in the following expression:

$$\dot{V} = \sum_{i=1}^{n} y_i \left\{ r_i(\underline{x}; \underline{y}) - r_i(\underline{x}; \underline{0}) \right\}. \qquad (8)$$

<u>Example 11, 8</u>   <u>Leighton's Exact Equations of First Order</u>

Consider the system defined by

$$\dot{x}_i = r_i(x_1, x_2, \ldots . x_n) = r_i(\underline{x}),$$   (1)

where the $r_i$ are of class $C^1$ in a neighborhood N of $\underline{x} = \underline{0}$, and the point $\underline{0}$ is an isolated equilibrium point of (1). Further, we assume that in N

$$\frac{\partial r_i}{\partial x_j} = \frac{\partial r_j}{\partial x_i} .$$   (2)

Because of (2), the following line integral is independent of path:

$$V = \int_{\underline{0}}^{\underline{x}} r_i (\underline{x}) \, dx_i$$   (i summed, 1 to n).   (3)

If V is computed along the path from $(0, 0, \ldots , 0)$ to $(x_1, 0, \ldots , 0)$ to $\ldots$ to $(x_1, x_2, \ldots , x_n)$, then we can easily see that

$$\dot{V} = \sum_{i=1}^{n} r_i (\underline{x}) \, r_i (\underline{x}) .$$   (4)

Thus, $\dot{V}$ is positive definite in N. If V is negative definite in N, the origin is asymptotically stable. In all other cases, by Liapunov's instability theorem, the origin is unstable.

## Special Case

Consider the system

$$\dot{x} = ax + by,$$

$$\dot{y} = bx + cy,$$

where we assume that the constants a, b, c satisfy the conditions $a + c < 0$, $ac - b^2 < 0$. Thus, the V-function becomes

$$V = \int_{\underline{0}}^{\underline{x}} r_1(\underline{x}) \, dx_1 + \int_{\underline{0}}^{\underline{x}} r_2 (\underline{x}) \, dx_2 ,$$

$$= \int_0^{\underline{x}} (ax)\ dx\ +\ \int_0^{\underline{x}} (bx\ +\ cy)\ dy\ ,$$

$$= \frac{1}{2} \left\{ ax^2\ +\ 2bxy\ +\ cy^2 \right\} ,$$

where

$$\dot{V} = (ax + by)^2\ +\ (bx + cy)^2\ .$$

Therefore, $\underline{0}$ is asymptotically stable for the conditions satisfied by a, b, c.

### Example 12, [10]    Skidmore's Fourth Order Case

In this example we consider the stability and instability of an isolated equilibrium point of a fourth-order autonomous system of the form

$$\dot{x} = y$$
$$\dot{y} = z$$
$$\dot{z} = w \tag{1}$$
$$\dot{w} = -w \not{\gamma}(x, y, z)\ -\ \ominus(x,\ y,\ z),$$

associated with the differential equation

$$\ddddot{x}\ +\ \not{\gamma}(x,\ \dot{x},\ \ddot{x})\ \dddot{x}\ +\ \ominus(x,\ \dot{x},\ \ddot{x})\ =\ 0. \tag{2}$$

We restrict $\not{\gamma}$ and $\ominus$ in the following ways:   $\ominus(0,\ 0,\ 0) = 0$,

$\not{\gamma}$ and $\ominus$ are of class $C^2$ near the origin. We also suppose that the origin is an isolated equilibrium point of the system (1).

In the following discussion, the subscript "0" denotes the quantity being evaluated at the origin and $\propto$ is a constant that will be assigned a particular value at a later time. In the analysis of the stability of the equilibrium solution the Liapunov function used by Skidmore is

$$2V = \left\{ \alpha \gamma_0 \frac{\partial \Theta_0}{\partial z} - \frac{\partial \Theta_0}{\partial x} \right\} y^2 + 2 \alpha \gamma_0^2 \, y \, z \; +$$

$$+ 2 \alpha \gamma_0 \int_0^x \Theta(x, 0, 0) \, dx + 2\alpha\gamma_0 \, yw \; +$$

$$\tag{3}$$

$$+ 2 (\gamma_0 - \alpha) \int_0^z \gamma(x, y, z) \, dz + 2 \gamma_0 \int_0^y \Theta(x, y, 0) \, dy \; +$$

$$+ 2 \gamma_0 \, z \, w + 2 \int_0^z \Theta(x, y, z) \, dz + w^2.$$

The time derivative of V along the trajectories of (1) is given by:

$$-\dot{V} = y^2 \left\{ \frac{\gamma_0}{y} \int_0^y \left\{ \alpha \frac{\partial \Theta(x, y, 0)}{\partial y} - \frac{\partial \Theta(x, y, 0)}{\partial x} \right\} \, dy \right\} +$$

$$+ z^2 \left\{ -\alpha \gamma_0^2 + \frac{1}{z} \int_0^z \left[ \gamma_0 \frac{\partial \Theta(x, y, z)}{\partial z} - \frac{\partial \Theta(x, y, z)}{\partial y} + (\alpha - \gamma_0) \frac{\partial \gamma(x, y, z)}{\partial y} z \right] dz \right\}$$

$$+ w^2 \left\{ \gamma(x, y, z) - \gamma_0 \right\} + 2yw \left\{ \frac{\alpha\gamma_0}{2} \right\} \left\{ \gamma(x, y, z) - \gamma_0 \right\} + \tag{4}$$

$$+ 2 \, wz \left\{ \frac{\alpha}{2} \right\} \left\{ \gamma(x, y, z) - \gamma_0 \right\} + 2yz \left\{ \frac{1}{2} \right\} \left\{ \frac{\partial \Theta_0}{\partial x} + \right.$$

$$- \alpha\gamma_0 \frac{\partial \Theta_0}{\partial z} + \frac{1}{z} \int_0^z \left\{ \alpha \gamma_0 \frac{\partial \Theta(x, y, z)}{\partial z} - \frac{\partial \Theta(x, y, z)}{\partial x} + \right.$$

$$+ (\alpha - \gamma_0) z \frac{\partial \gamma(x, y, z)}{\partial x} \left. \right\} \, dz \left. \right\} .$$

The author uses the usual theorems relating the properties of the V-function with the local stability properties of the origin.

Some of the restrictions which are placed on $\Theta$ and $\gamma$ such that V is locally positive definite in a neighborhood of the origin are

$$\frac{\partial \Theta_0}{\partial x} > 0 \;, \quad \frac{\partial \Theta_0}{\partial y} > 0 \;, \quad \frac{\partial \Theta_0}{\partial z} > 0 \;, \; \gamma_0 > 0 \;,$$

$$\frac{\partial \Theta_0}{\partial z} \gamma_0 - \frac{\partial \Theta_0}{\partial y} > 0 \;, \tag{5}$$

$$\frac{\partial \Theta_0}{\partial y} \frac{\partial \Theta_0}{\partial z} \gamma_0 - \frac{\partial \Theta_0}{\partial x} \gamma_0^2 - \left\{ \frac{\partial \Theta_0}{\partial y} \right\}^2 > 0 \;,$$

where $\alpha$ is taken to be

$$\alpha = \left\{ \frac{\partial \Theta_0}{\partial x} \Big/ \frac{\partial \Theta_0}{\partial y} \right\}.$$

We <u>note</u> in passing that the system (1) has equilibrium points when $y = z = w = 0$, $\Theta(x, 0, 0) = 0$. These points are also critical points of the Liapunov function in equation (3). But the author in [10] only is concerned with the equilibrium point at the origin.

From equation (4) we see that $-\dot{V}$ is a quadratic form in $[y, z, w]$ with variable coefficients which are functions of $y$, $z$, $w$; that is,

$$-\dot{V} = [y, z, w] \; \underline{A} \begin{bmatrix} y \\ z \\ w \end{bmatrix} \;, \tag{6}$$

where the elements of $\underline{A}$ are the terms in the brackets in equation (4). The form $-\dot{V}$ will be positive semidefinite when the principal minors $M_1$, $M_2$, $M_3$ of matrix $\underline{A}$ satisfy the inequalities:

$$M_1 > 0, \; M_2 > 0, \; M_3 > 0 \tag{7}$$

in a neighborhood of the origin. Therefore, the following stability results are obtained for the system in (1) based upon the inequalities in (7).

## Theorem 1

H)   If the following conditions hold in a neighborhood of the origin:

(i)   $\gamma (x, y, z)$ and $\Theta (x, y, z)$ are of class $C^2$;

(ii)   $\Theta (0, 0, 0) = 0$;

(iii)  conditions in (5) and (7) hold;

(iv)  $\dot{V} \not\equiv 0$ along every nontrivial solution of (1),

C)  then the origin of (1) is an asymptotically stable equilibrium point.

Skidmore studied the instability of the origin by employing the following theorem of Krasovskii,  [11, p. 69] .

Theorem 2

H)  If there exists a bounded neighborhood N of the origin, a region $N_1$ contained in N, and a scalar function $V(\underline{x})$ such that

(i)  $V(\underline{x})$ is of class $C^1$ in N;

(ii)  $\underline{0}$ belongs to the boundary of $N_1$; the boundaries of N and $N_1$ have points in common;

(iii)  $V(\underline{x}) > 0$ for $\underline{x}$ in $N_1$; $V(\underline{x}) = 0$ for $\underline{x}$'s belonging to the boundary of $N_1$ but not to the boundary of N;

(iv)  $\dot{V}(\underline{x}) \geqslant 0$ for x belonging to $N_1$;

(v)  the set R, which contains all the $\underline{x}$'s in $N_1$ for which $\dot{V}(\underline{x}) = 0$, does not contain any positive invariant set of the system

$$\dot{\underline{x}} = \underline{f}(\underline{x}), \quad \underline{f}(\underline{0}) = \underline{0};$$

C)  then the equilibrium point at $\underline{0}$ is unstable.  ($\underline{x}$ is an n-vector in this theorem.)

Skidmore applied Theorems 1 and 2 to several special cases.  These special cases will now be presented as Examples 13, 14, 15, 16 and 17.

Example 13, [10]    Fourth-Order Linear System

Consider the linear system defined by

$$\ddddot{x} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = 0,$$

where the associated system is

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = w$$

$$\dot{w} = -a_1 w - a_2 z - a_3 y - a_4 x.$$

The conditions given in (5) in Example 12 are equivalent to the Routh-Hurwitz conditions:

$$A_1 = a_1 > 0$$

$$A_2 = a_1 a_2 - a_3 > 0$$

$$A_3 = a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0,$$

$$A_4 = a_1 a_4 A_3 > 0.$$

If $\alpha = a_4/a_3$, then V and $\dot{V}$ as defined in Example 12 verify that the origin is completely stable.

For $a_i > 0$ (i = 1, 2, 3, 4) and $A_2 < 0$, we have that $A_3 < 0$. Let $N_1$ in Theorem 2, in Example 12, be the points (x, 0, 0, w). The $\dot{V}$ given above is $-\left\{A_3/a_3\right\}^2 z^2$, but the corresponding surface z = 0 contains no invariant points of the linear system. Thus, from Krasovskii's Theorem, the origin is unstable.

Therefore, for the linear case, Skidmore's results are consistent with other methods of stability analysis.

## Example 14, [10]    Fourth Order Analogue of an Example by Szegö

The conditions in (5), in Example 12, are sufficient for V to be positive definite locally but are not necessary as can be seen in the following example:

$$\ddddot{x} + 6\dddot{x} + \ddot{x} + 6\dot{x} + 4x^3 = 0.$$

The state variable notation for this system is given as

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = w$$

$$\dot{w} = -6w - 11z - 6y - 4x^3.$$

The linear approximation of this system about the origin yields the character-istic roots 0, -1, -2, -3. Therefore, the asymptotic stability of the origin can not be predicted by the linear approximation and thus Liapunov's second method is required. In this particular example, the expression for $\dfrac{\partial \mathcal{O}_0}{\partial x}$

is zero and not greater than zero as required by Skidmore's conditions. But if $\alpha = 4/3$ , the V-function in Example 12 becomes

$$2V = 16x^4 + 124y^2 + 39z^2 + w^2 + 48x^3 y + 8x^3 z +$$
$$+ 108yz + 16yw + 12zw =$$

$$= (w + 6z + 8y)^2 + \frac{1}{3} (3z + 4x^3 + 6y)^2 + \frac{16}{3} (3y + x^3)^2 +$$

$$+ \frac{16}{3} x^4 (3-2x^2);$$

so that V is positive definite when $x^2 < 3/2$. The time derivative of V is given by

$$-\dot{V} = 3 (x^2 y - 2z)^2 - 3 y^2 (x^4 - 24x^2 - 16),$$

where $-\dot{V}$ is positive semidefinite when $x^2 < \sqrt{160} - 12$ .

Hence, the origin is locally asymptotically stable, even though not all of the conditions in (5) in Example 12 are satisfied.

Skidmore investigated the region of asymptotic stability for this system by using a theorem due to Leighton [12].

## Theorem

"Suppose the system $\dot{\underline{x}} = \underline{f}(\underline{x})$ has an isolated equilibrium point at the origin, and suppose further that there exists a function $V(\underline{x})$ of class $C^2$

in $E_n$ which is locally positive definite around $\underline{x} = \underline{0}$, with $V(\underline{0})$, with $V(\underline{0}) = 0$.

Suppose further that V has at most a finite number of critical points, and that $\dot{V} \leq 0$ in $E_n$, and that the origin is the only point in the invariant set for which $V = 0$. It follows that $\underline{0}$ is an asymptotically stable equilibrium point of $\underline{\dot{x}} = \underline{f}(\underline{x})$ and regions of asymptotic stability of $\underline{0}$ are bounded by surfaces defined by $V(\underline{x}) = a$, for each a on an interval $0 < a < K$.

The number K is a positive critical point if V, if V has at least two critical points; otherwise $K = \displaystyle\lim_{\underline{x} \longrightarrow \infty} \inf \{V(\underline{x})\}$.

In the example considered here the critical points of V are $P_0(0, 0, 0, 0)$, $P_1(1, -1/3, -2/3, 20/3)$, $P_2(-1, 1/3, 2/3, -20/3)$, and the corresponding critical values are $2V = 0$, $16/3$, $16/3$. Thus, the domain of asymptotic stability of the origin is bounded by the surface $V = 8/3$, where V is defined in the above discussion. But in this particular case $\dot{V} \leq 0$ further restricts the region, as mentioned previously.

In a similar analysis and by using Krasovskii's Theorem, Skidmore proves that the origin of the following system is unstable:

$$\ddddot{x} + 6\dddot{x} + 11\ddot{x} + 6\dot{x} - 4x^3 = 0.$$

## Example 15, [10]  Skidmore's Theory Applied to Cartwright's Example

In reference [13], Cartwright studied the asymptotic stability of the origin for the system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w$$
$$\dot{w} = -a_1 w - a_2 z - a_3 y - f(x), \tag{1}$$

associated with the differential equation

$$\ddddot{x} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + f(x) = 0$$

where $a_1$, $a_2$, $a_3$ are constants and $f(0) = 0$. This equation is a

special case of Skidmore's equation, Example 12, where $\gamma = a_1$ and $\Theta = a_2 z + a_3 y + f(x)$.

Cartwright derived the Liapunov function given by

$$2V_c = a_1^2 a_3 \left\{ W + a_1 z + \left(a_2 - \frac{a_3}{a_1}\right) y \right\}^2 +$$

$$+ a_3^2 a_1 \left\{ z + a_1 y + \frac{a_1}{a_3} f(x) \right\}^2 + a_3 \left\{ a_1 a_2 a_3 - a_3^2 - a_1^2 f'(x) \right\} y^2 + \qquad (2)$$

$$+ 2a_1 \int_0^x f(x) \left\{ a_1 a_2 a_3 - a_3^2 - a_1^2 f'(x) \right\} dx ,$$

where

$$- \dot{V}_c = a_1 a_3 \left\{ a_1 a_2 a_3 - a_3^2 - a_1^2 f'(x) + \frac{1}{2} a_1 y f''(x) \right\} y^2 .$$

The conclusions drawn by Cartwright from $V_c$ and $\dot{V}_c$ concerning the stability of $\underline{O}$ are as follows. For every $V_0 > 0$, there is a domain $D_0$ of asymptotic stability of $\underline{O}$ given by $V_c (x, y, z, w) < V_0$, provided the following conditions hold in $D_0$:

(a) $f'(x) > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$;

(b) $a_1 a_2 - a_3 > 0$;

(c) $a_1 a_2 a_3 - a_3^2 - a_1^2 f'(x) \geqslant \delta > 0$;

(d) $f''(x)$ continuous;

(e) $\left| f''(x) \right| y < \delta / a_1$ ;

(f) $\int_0^x f(x) dx \longrightarrow \infty$ as $|x| \longrightarrow \infty$ .

Skidmore observed that if $f''(x) \neq 0$, then "f" continuous" fails, in general, to hold throughout $E_4$; so that complete asymptotic stability of $\underline{O}$ can <u>not</u> be determined from $V_c$ in equation (2). But if Skidmore's

V is used, one can conclude that $\underline{0}$ is completely stable under appropriate conditions.

    <u>First</u>, Skidmore considers local asymptotic stability. The V given in Example 12 and the value of $\alpha$ are given by

$$\alpha = f'(0)/a_3,$$

$$2V = \frac{2a_1}{a_3} \int_0^x f(x) \left\{ f'(0) - f'(x) \right\} dx +$$

$$+ \frac{f'(0)}{a_3^2} \left\{ a_1 a_2 a_3 - a_3^2 - a_1^2 f'(0) \right\} y^2 + \tag{3}$$

$$+ \frac{1}{a_1 a_3} \left\{ a_1 a_2 a_3 - a_3^2 - a_1^2 f'(0) \right\} z^2 +$$

$$+ \left\{ w + a_1 z + \frac{a_1}{a_3} f'(0) y \right\}^2 + a_1 a_3 \left\{ y + \frac{f(x)}{a_3} + \frac{z}{a_1} \right\}^2 ,$$

where

$$- \dot{V} = a_1 \left\{ f'(0) - f'(x) \right\} y^2 + \left\{ f'(0) - f'(x) \right\} y z +$$

$$+ \frac{1}{a_3} \left\{ a_1 a_2 a_3 - a_3^2 - a_1^2 f'(0) \right\} z^2 .$$

The conditions (5) and (7) in Example 12 become

    (a)'. $f'(0) > 0, \ a_1 > 0, \ a_2 > 0, \ a_3 > 0;$

    (b)' $a_1 a_2 - a_3 > 0;$

    (c)' $A_3 = a_1 a_2 a_3 - a_3^2 - a_1^2 f'(0) > 0;$

    (d)' $f'(0) > f'(x) > f'(0) - \dfrac{4a_1}{a_3} A_3;$

Thus V and $-\dot{V}$ are <u>locally</u> positive definite and positive semidefinite, respectively. Also, since the surfaces $x = z = 0, \ y = z = 0$ do <u>not</u> contain

invariant points of the system, the origin is <u>locally</u> asymptotically stable.
(It can be shown that locally the conditions (a)' ___ (d)' are more general
than Cartwright's conditions.)

The origin of the system in (1) can be shown to be <u>completely</u> asymptot-
ically stable if conditions (a)' ___ (d)' hold and if, in addition,

(e)' the only critical point of V is $\underline{0}$;

(f) $\lim\limits_{\|\underline{x}\| \longrightarrow \infty} \left\{ \inf V(\underline{x}) \right\} = + \infty$, where $\|\underline{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$ ,

hold. These conditions (e)' and (f)' have been weakened in $|12|$.

## Example 16, [10]     Special Case of Example 15

In this particular example, Skidmore's V-function yields complete asymptotic
stability of the origin, whereas Cartwright's V-function does not. The
equation we consider is

$$\dddot{x} + \dddot{x} + 3\ddot{x} + \dot{x} + \arctan x = 0,$$

and the corresponding system is

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w,$$

$$\dot{w} = - w - 3z - y - \arctan x.$$

Here, $a_1 = 1$, $a_2 = 3$, $a_3 = 1$, and $f(x) = \arctan x$, where
$- \pi/2 < \arctan x < \pi/2$ . Therefore conditions (a)' ___ (d)'
in Example 15 are satisfied for all x. The V and $\dot{V}$ functions of Skidmore are

$$2V = 2 \int_0^x \left\{ 1 - \frac{1}{1 + x^2} \right\} \arctan x \ dx + y^2 + z^2 +$$

$$+ (y + z + w)^2 + (\arctan x + y + z)^2 ,$$

and

$$-\dot{V} = \left\{1 - \frac{1}{1+x^2}\right\} y^2 + \left\{1 - \frac{1}{1+x^2}\right\} yz + z^2 =$$

$$= \frac{1}{4}\left\{2z + \frac{x^2 y}{1+x^2}\right\}^2 + \frac{1}{4}\left\{\frac{x^2}{1+x^2}\right\}\left\{\frac{3x^2+4}{1+x^2}\right\} y^2.$$

The V given above has its only critical point at the origin and because

$$\int_0^x \left\{1 - \frac{1}{1+x^2}\right\} \arctan x \, dx \longrightarrow \infty \quad \text{as} \quad |x| \longrightarrow \infty,$$

conditions (e)' and (f)' are satisfied. Thus, the origin is completely as asymptotically stable. Cartwright's conditions will not yield complete asymptotic stability of the origin.

Example 17, [10]    Skidmore's Results Applied to Ezeilo's Example

In reference [5], Ezeilo considered the equation

$$\ddddot{x} + a_1 \dddot{x} + a_2 \ddot{x} + g(\dot{x}) + a_4 x = 0,$$

with the corresponding system defined by

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w,$$

$$\dot{w} = -a_1 w - a_2 z - g(y) - a_4 x.$$

This is a special case of Skidmore's fourth order differential equation where $\psi = a_1$ and $\Theta = a_2 x + g(y) + a_4 x$.

Ezeilo showed that the origin is completely asymptotically stable if the following conditions are satisfied:

(a)  $a_1 > 0, a_2 > 0, a_4 > 0;$

(b)  $g(0) = 0, \dfrac{g(y)}{y} \geqq a_3 > 0 \ (y \neq 0);$

(c)  $g'(y)$ continuous and $g'(y) \leqq A_3$ for all y, where

$$a_1 \, a_2 \, a_3 - A_3 \, a_3 - a_1^2 \, a_4 > 0, \text{ and}$$

$$A_3 = a_1 \, a_2 \, a_3 - a_3^2 - a_1^2 \, f'(0) > 0;$$

(d)- $\left[ g\ (y)\big/y - g'(y) \right] \leq \delta_1$      , for $y \neq 0$, where $\delta_1$ is any

constant such that

$$\delta_1 \ < \ \frac{2a_4\ (a_1a_2a_3 - a_3^2 - a_1^2\ a_4)}{a_1\ a_3^2} \ .$$

The conditions which must be satisfied in Skidmore's Theorem are

(a)'   $a_1,\ a_2,\ a_4,\ g'\ (0)\ >\ 0;$

(b)'   $a_1\ a_2 - g'(0)\ >\ 0;$

(c)'   $a_1a_2g'(0) - a_1^2\ a_4 - \left\{g'(0)\right\}^2\ >\ 0;$

(d)'   $g(y)\big/y\ -\ g'\ (0)\ >\ 0;$

(e)'   $a_1\ a_2\ g'(0)\ -\ a_1^2\ a_4 - g'(0)\ g'\ (y)\ >\ 0.$

For $\alpha = a_4\big/g'(0)$      , the Liapunov function

$$2V = \left\{\frac{a_1a_4^2}{g'(0)}\right\}x^2 + \left\{\frac{a_1a_2a_4}{g'(0)} - a_4\right\}y^2 + 2a_1 \int_0^y g(y)\ dy\ +$$

$$+ \left\{a_1^2 + a_2 - \frac{a_1a_4}{g'(0)}\right\}z^2 + w^2 + 2a_1a_4\ xy + 2a_4x\,z\ +$$

$$+ \left\{\frac{2a_1^2a_4}{g'(0)}\right\}yz + 2\ g(y)z + \left\{\frac{2a_1a_4}{g'(0)}\right\}\ yw + 2a_1\,z\,w,$$

where

$$-\dot{V} = \frac{a_1a_4}{g'(0)} \left\{\frac{g(y)}{y} - g'\ (0)\right\}y^2\ +$$

$$+\ \frac{1}{g'(0)}\left\{a_1a_2\ g'(0) - a_1^2\ a_4 - g'(0)\ g'\ (y)\right\}z^2\ .$$

Under conditions (a)' ___ (c)', V is positive definite and conditions (d)', (e)'

imply that $-\dot{V}$ is positive semidefinite in the neighborhood of the origin.

Further, the surface $y = z = 0$, where $\dot{V} = 0$, contains no invariant points

of the system. The conclusion is that the origin is locally asymptotically

stable. If we assume that Ezeilo's conditions (a) ___ (d) hold and take

$\propto = \begin{Bmatrix} a_4 \\ a_3 \end{Bmatrix}$ in the above V-function of Skidmore, then this $V = a_1$ $(V_E)$,

$V_E$ being Ezeilo's V-function. Therefore, the origin can be shown to be

completely asymptotically stable.

The _next set of examples_ are from a summary of third and fourth order equations

collected by G. Sansone in reference [14]. Some of the references given by

Sansone were unavailable (as far as we are concerned), so that some of the dis-

cussion which follows will be brief.

## Example 18, [15] Šimanov's Example

The system is defined by the following third order equation:

$$\dddot{x} + f(x, \dot{x})\, \ddot{x} + b\dot{x} + cx = 0,$$

where b and c are constants. The result of Šimanov's work is:

the origin of this system is globally asymptotically stable if

(1) (0, 0) is an isolated equilibrium solution of the system;

(2) $b > 0$, $c > 0$;

(3) $f(x, \dot{x}) > c/b$, for all $x, \dot{x}$ ;

(4) $\dot{x}\dfrac{\partial f(x, 0)}{\partial x} < 0$, for all $x, \dot{x}$.

## Example 19, [16] Krasovskii's Example

In reference [16], Krasovskii gives necessary and sufficient conditions for the

asymptotic stability of the origin of the following system:

$$\dot{x}_1 = f_1(x_1) + a_1 x_2 + b_1 x_3,$$

$$\dot{x}_2 = f_2(x_2) + a_2 x_2 + b_2 x_3,$$

$$\dot{x}_3 = f_3(x_3) + a_3 x_2 + b_3 x_3,$$

where $\left|f_i(x)\right| \leq M(x)$ for sufficiently small $|x|$ , assuming $a_1 b_1 \neq 0$.

## Example 20, [17, 18, 19]   Tusov's Example

Tusov consider the third order system defined by

$$\dot{x}_1 = \sum_{K=1}^{3} a_{1K} x_K + f(x_1)$$

$$\dot{x}_2 = \sum_{K=1}^{3} a_{2K} x_K ,$$

$$\dot{x}_3 = \sum_{K=1}^{3} a_{3K} x_K ,$$

where $a_{ik}$ are real and f satisfies the usual existence and unicity conditions in the whole space. Tusov derives the sufficient conditions for the origin to be asymptotically stable. Furthermore these conditions of stability dictate that f must satisfy the inequalities

$$\alpha x_1^2 < x_1 f(x_1) < \beta x_1^2 ,$$

where $\alpha$ and $\beta$ are the extreme values of the parameter "a", such that on replacing $f(x_1)$ by $ax_1$ in the above system, the characteristic roots of the corresponding linear system have negative real parts.

## Example 21, [1, 20, 21, 22]   Pliss's Examples

In reference [20], Pliss considers the system defined by

$$\dot{x} = y - f(x) ,$$

$$\dot{y} = z - x ,$$

$$\dot{z} = - x ,$$

where $f(0) = 0$, $xf(x) > x^2$ for $x \neq 0$. Pliss derives sufficient conditions for the stability of the origin "in the large." He also derives sufficient conditions for the existence of periodic solutions.

In reference [1] he considers the system

$$\dot{x} = y - f(x) ,$$

$$\dot{y} = z - x ,$$

$$\dot{z} = -ax - bf(x),$$

where f is Lipschitzian, $f(0) = 0$, $f(x)/x > x + bf(x)/x$ for $x > 0$. Pliss gives sufficient conditions under which this system is stable in the large. Furthermore, he proves that periodic solutions exist if

(1) $a > 0, 0 \leq b < 1, a + b \geq 1$;

(2) $hx \leq f(x) - \left\{\dfrac{ax}{1-b}\right\}$ if $0 \leq x \leq x_1$;

(3) $0 < f(x) - \left\{\dfrac{ax}{1-b}\right\} < \lambda$ for $x \geq x_2$,

where $h > \dfrac{1-b}{a}$ ;

(4) $\lambda$ , $x_1$, $x_2 - x_1$ are sufficiently small positive numbers.

In reference [21], Pliss gives necessary and sufficient conditions for stability in the large of the system

$$\dot{x} = y - ax - f(x) ,$$

$$\dot{y} = x - bf(x) ,$$

$$\dot{z} = -c f(x).$$

The conditions are

(1) $ab > c, b > 0, c > 0, f(0) = 0$;

(2) $xf(x) > 0$ for $x \neq 0$;

(3) $\overline{\lim_{x \to -\infty}} \left\{ f(x) - \int_0^x f(s)ds \right\} = \infty$

(4) $\overline{\lim_{x \to -\infty}} \left\{ -f(x) - \int_0^x f(s) ds \right\} = \infty.$

Finally in reference [22], Pliss gives, without proof, many conditions under which all the solutions of a nonlinear system of the form

$$\dot{\underline{x}} = \underline{A} \ \underline{x} + f(x_1) \ \underline{b} \ , \ \underline{x}(0) = \underline{c} \ ,$$

approach zero as t $\longrightarrow \infty$. Here, $\underline{x}$ is a 3-dimensional vector, $\underline{A}$ is a constant matrix, and $f(x_1)$ is a scalar function.

<u>Example 22, [23]</u>     <u>Vaisbord- Boundedness of Solution</u>

In reference [23], Vaisbord considers the system

$$\dot{x} = f_{11}(x) + f_{12}(y) \ ,$$

$$\dot{y} = f_{23}(z) \ ,$$

$$\dot{z} = f_{31}(x) + f_{32}(y) + f_{33}(z) \ .$$

The existence and continuity of the $f_{ij}$'s are assumed and the following hypotheses are satisfied:

(1)  $f_{ij}(0) = 0$ ;

(2)  $f'_{11}(x) < 0, \ \lim\limits_{|x| \to \infty} f_{i1}(x) = - \infty$       $(i = 1, 3)$ ;

(3)  $f_{12}(y) < c, \ f'_{23}(z) > 0, \ \left| f_{23}(z) \right| < a \ |z|$ ;

and when  $\lim\limits_{y \to \infty} \left| f_{32}(y) \right| < \infty, \quad \lim\limits_{y \to -\infty} \left| f_{32}(y) \right| < \infty,$

then  $\lim\limits_{y \to \infty} f'_{32}(y) = 0, \quad \lim\limits_{y \to -\infty} f'_{32}(y) = 0,$ respectively;

(4)  on the solution curves of

$$f_{31}(x) + f_{32}(y) = 0 \ ,$$

we have

$$\left\{ f'_{21}(y) - f'_{11}(x) \right\} \bigg/ \left\{ f'_{31}(x) - f'_{32}(y) \right\} \quad > \quad N > 0 \ ,$$

$$0 \quad < \quad A \quad < \quad f'_{11}(y) \bigg/ f'_{31}(x) \ ;$$

(5)  $ca \big/ AN^2 \quad < \quad 1$ ;

(6) $f'_{23}(0) \left\{ f'_{33}(0) \ f'_{32}(0) + f'_{31}(0) \ f'_{12}(0) \right\} +$

$- f'_{11}(0) \ f'_{33}(0) \left\{ f'_{11}(0) - f'_{33}(0) \right\} < 0$.

Under these hypotheses the above system has a periodic solution; and all solutions

of the system are <u>bounded in the large</u> as t $\longrightarrow \infty$.

An outline of the procedure followed by Vaisbord in his analysis is as follows:

(1) linearize the system about the origin, the linearized system being

$$\dot{\xi} = f'_{11}(0) \ \xi + f'_{12}(0) \ \eta \ ,$$

$$\dot{\eta} = f'_{23}(0) \ \zeta \ ,$$

$$\dot{\zeta} = f'_{31}(0) \ \xi + f'_{32}(0) \ \eta + f'_{33}(0) \ \zeta \ ;$$

(2) the characteristic roots of this linear system are such that one root has a

negative real part and two roots have positive real parts;

(3) by a <u>fixed - point theorem</u> the existence of a periodic solution is proved;

(4) from topological considerations the boundedness of the solutions is proved.

<u>Example 23, [24]</u>   <u>Ogurcov's Examples</u>

Ogurcov studies the <u>asymptotic stability in the large</u> of the equilibrium solutions

corresponding to four different autonomous systems.  Liapunov's second method was

used in the study, and what follows is a summary of the stability results.

(A) $\dddot{x} + \mathcal{Y}(x, \dot{x}) \ \ddot{x} + \phi(\dot{x}) + f(x) = 0$

The stability results, which are sufficient conditions, are as follows:

(1) if $F(x, y) = 2 \propto \int_{0}^{x} f(\mathcal{S}) \ ds + f(x) \ y + \int_{0}^{y} \phi(s) \ ds$,

where $\propto$ is a positive constant and $y = \dot{x}$, then $F \longrightarrow \infty$ as $x^2 + y^2 \longrightarrow \infty$ ;

(2) $\phi(0) = f(0) = 0$, $f(x)/x > 0$, $\mathcal{Y}(x, y) \geqslant 2 \propto$;

(3) $\quad x \dfrac{\partial \gamma(x, y)}{\partial x} \leq 0$ ;

(4) $\quad 2\alpha \phi(y)/y \quad - \quad f'(x) \quad > \quad 0$;

or

(5) $\quad 2\alpha \phi(y)/y \quad - \quad f'(x) \quad - \alpha^2 \left\{ \gamma(x, y) - 2\alpha \right\} \quad > 0.$

(B) $\quad \dddot{x} + a\ddot{x} + \phi(x) \dot{x} + bx = 0$

The sufficient conditions for stability are:

(1) $\quad a > 0, \ b > 0, \ \phi(x) > b/a$ ;

(2) $\quad \displaystyle\int_0^x ds \left\{ \int_0^s \left\{ \phi(\tau) - b/a \right\} d\tau \right\} \longrightarrow \infty$ as $x \longrightarrow \infty$.

(C) $\quad \dddot{x} + \gamma(\dot{x}, \ddot{x}) \ddot{x} + c\ddot{x} + b\dot{x} + ax = 0$

The values a, b, c are constants. The sufficient conditions for stability are:

(1) $\quad a > 0, \ b > 0, \ \gamma(y, z) > 0$, where $y = \dot{x}, \ z = \ddot{x}$;

(2) $\quad bc \gamma(y, z) - b^2 - a^2 \gamma(y, z) > 0$ ;

(3) $\quad z \dfrac{\partial \gamma(y, z)}{\partial y} \leq 0.$

(D) $\quad \ddddot{x} + d\dddot{x} + c\ddot{x} + \phi(\dot{x}) + ax = 0$

The stability conditions are:

(1) $\quad$ a, c, d are constants and $a > 0, \ d > 0$;

(2) $\quad \phi(0) = 0, \ \phi(y)/y > 0$ ;

(3) $\quad cd \ \phi(y)/y - \left\{ \phi(y)/y \right\}^2 - ad^2 > 0.$

## Example 24, [25]   Tabueva's Example

In reference [25], Tabueva derived conditions for the stability and existence of a periodic solution for the system defined by

$$\dddot{x} + a\ddot{x} + b\dot{x} + \sin x = e(t),$$

where a and b are constants. The stability in this case is asymptotic stability.

Example 25, [26, 27, 28, 29, 30, 31]    E z eilo's Examples

In the five papers, [26] to [30], Ezeilo studies the following third order

nonautonomous system:

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t) ,$$

where a, b are positive constants.  The findings of Ezeilo are:

(1)   sufficient conditions for boundedness of solutions;

(2)   when p(t) is a continuous periodic function of t with a least period $T > 0$,

   then sufficient conditions for the existence of periodic solutions with periods (nT),

   $n \geqslant 1$ and n an integer, are given;

(3)   sufficient conditions for the existence of at least one solution of the system

   with period T.

In reference [31] , Ezeilo gives an existence theorem for a solution of

$$\dddot{x} + f(x, \dot{x}) \ddot{x} + g(\dot{x}) + h(x) = p(t) ,$$

where $x_0 = x(0)$, $y_0 = \dot{x}(0)$, $z_0 = \ddot{x}(0)$, for $t \geqslant 0$.

Example 26   [32]    Pliss's Example

In reference [32], Pliss gives a generalization of the results of Ezeilo for

the system defined by

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = P(x,\dot{x},\ddot{x},t).$$
$$\dddot{y} + a\ddot{y} + \phi(\dot{y}) + y = G(y,\dot{y},\ddot{y},t),$$
$$\ddot{z} + g(\ddot{z}) + \dot{z} + a z = Q(z,\dot{z},\ddot{z},t) .$$

In the case where P, G, Q are periodic with respect to t, Brouwer's fixed point

theorem is used to prove the existence of a periodic solution.

Example 27, [33, 34, 35, 36]    Skačkov's Work

The system which Skačkov studied is defined by:

$$\dot{x} = ax + b y + f(x, y, z) ,$$
$$\dot{y} = cx + dy + g(x, y, z) ,$$
$$\dot{z} = h (x, y, z) ,$$

where a, b, c, d are constants such that ad - bc $\neq$ 0, and where f, g, h are power series with real coefficients beginning with terms of at least degree two. The investigations conducted by Skačkov were as follows:

(1)  the behavior of the integral curves,

(2)  stability in the large of equilibrium solutions,

(3)  existence and character of the system's singular points; that is, the behavior of the characteristic curves in the neighborhood of the singular points.

This completes the summary given by G. Sansone in reference [14], concerning stability theory.

## Example 28, [37, 38, 39, 40, 41]    Duffin's Work

In reference [37], Duffin determines when a certain system of nonlinear differential equations has a unique asymptotic solution. That is, to determine under what conditions will all the solutions of the system approach each other as the independent variable becomes infinite. The physical application of the system under investigation is in the field of electrical vibrations in networks.

## Definition 1

A linear network is a collection of linear inductors, linear resistors, and linear capacitors arbitrarily interconnected.

## Definition 2

A quasi-linear resistor is a conductor whose differential resistance lies between positive limits.

The main theorem in reference [37] can be paraphrased in the following way: "If in a linear network, the linear resistors are replaced by quasi-linear resistors, then, as in the linear case, after sufficient time has elapsed there is a unique relation between the impressed force and the response."

The system being considered is an n - degree of freedom system. When the system is linear, the network equation is

$$\underline{L}\,\ddot{\underline{q}} + \underline{R}\,\dot{\underline{q}} + \underline{S}\,\underline{q} = \underline{e} ,$$ (1)

where

$\underline{L}$, $\underline{R}$, $\underline{S}$ ≡ constant, symmetric, positive

semi-definite matrices,

$\underline{y} \equiv \dot{\underline{q}} \equiv$ n-th order current vector,

$\underline{q}$ ≡ n-th order electric charge vector,

$\underline{e}$ ≡ n-th order electromotive force vector.

## Definition 3

A continuous vector function $\underline{V}(\underline{y})$ is a quasi-linear replacement of $\underline{R}\,\underline{y}$ provided

$$\underline{V}\,(\underline{y}_1) - \underline{V}\,(\underline{y}_2) = \underline{V}^* \cdot (\underline{y}_1 - \underline{y}_2) ,$$ (2)

where $\underline{V}^*$ is the symmetric Jacobian matrix which satisfies

$$(1/a)\,\underline{y}_T\,\underline{R}\,\underline{y} \leq \underline{y}_T\,\underline{V}^*\,\underline{y} \leq a\,\underline{y}_T\,\underline{R}\,\underline{y},$$ (3)

for some positive constant "a" independent of the vectors $\underline{y}_1$, $\underline{y}_2$, $\underline{y}$ .

## Main Theorem [37]

H) If (i) all solutions of (1) satisfy, $\dot{\underline{q}} \longrightarrow \underline{0}$ as $t \longrightarrow \infty$,

(ii) $\underline{V}$ is a quasi-linear replacement of $\underline{R}\,\underline{y}$,

(iii) for $t \geqslant 0$, the vectors $\underline{q}$ and $\underline{e}$ satisfy

$$\underline{L}\,\ddot{\underline{q}} + \underline{V}\,(\dot{\underline{q}}) + \underline{S}\,\underline{q} = \underline{e} ,$$ (4)

(iv) $\underline{q}^*$ is any other solution of (4),

(v) $\dot{\underline{q}}$ and $\dot{\underline{q}}^*$ are continuous,

C) then

$$\int_0^\infty \|\dot{\underline{q}} - \dot{\underline{q}}^*\|\,dt < \infty .$$ (5)

## Proof

Let $\underline{w} = \underline{q} - \underline{q}^*$. Then, from (2) and (4) we obtain the following equation for W:

$$\underline{L} \, \ddot{\underline{W}} + \underline{V}^* \, \dot{\underline{W}} + \underline{S} \, \underline{W} = \underline{0}. \tag{6}$$

Premultiple (6) by $\dot{\underline{W}}_T$ and use the symmetry properties of $\underline{L}$ and $\underline{S}$ to obtain

$$\frac{d}{dt} \left\{ \dot{\underline{W}}_T \underline{L} \, \dot{\underline{W}} + \underline{W}_T \, \underline{S} \, \underline{W} \right\} = -2 \, \dot{\underline{W}}_T \underline{V}^* \, \dot{\underline{W}}. \tag{7}$$

Integrating (7) gives

$$-\left\{ \dot{\underline{W}}_T \, \underline{L} \, \dot{\underline{W}} + \underline{W}_T \, \underline{S} \, \underline{W} \right\} + A = 2 \int_0^t \dot{\underline{W}}_T \, \underline{V}^* \, \dot{\underline{W}} \, dt \,.$$

Since $\underline{L}$ and $\underline{S}$ are positive semi-definite,

$$\int_0^\infty \dot{\underline{W}}_T \, \underline{V}^* \, \dot{\underline{W}} \, dt < A/2 \,. \tag{8}$$

Then from the properties of symmetric semi-positive matrices and the inequality (8), we can prove that (5) is satisfied.

## Notes About the Theorem

(1) This theorem says that $\dot{\underline{q}}$ and $\dot{\underline{q}}^*$ approach each other in-the-mean, but not necessarily pointwise.

(2) Since $\underline{L}$ and $\underline{S}$ are positive semi-definite and considering the equations (3) and (7), we see that $\left\{ \dot{\underline{W}}_T \, \underline{L} \, \dot{\underline{W}} + \underline{W}_T \, \underline{S} \, \underline{W} \right\}$ is playing "very nearly" the role of a Liapunov function.

In reference [38], Duffin shows that a network of quasi-linear conductors possesses a stable set of currents and proves that the stable set is unique. The criteria which this set must satisfy are the conservation of electricity and the single-valuedness of the electrical potential. In the proof of the statements, Duffin uses an analogy between elastic and electric networks.

In reference [39] , Duffin considers the quasi-linear properties of a certain n-dimensional transformation, $\underline{y} = \underline{S} \ (\underline{x})$. More precisely, he considers the sufficient conditions to guarantee the existence of a unique inverse. (This transformation is, of course, directly connected with his theory of electrical networks.)

In reference [40] , Duffin proves that a certain nonlinear system has one and only one periodic solution. The nonlinear network is obtained from a linear network by replacing linear resistors by quasi-linear resistors. The <u>central mathematical idea</u> explored in reference [40] is the treatment of the network equations as a transformation between Hilbert spaces. That is, the network equations define a transformation from the "Hilbert space of electric charge" to the "Hilbert space of electromotive force". Duffin specifies the sufficient conditions which must be imposed on this nonlinear transformation in order that the inverse transformation exists.

The theorem and proof concerning the existence of a unique periodic solution for this nonlinear system in [40] depends on the above mathematical concepts and is very similar to the work of reference [37] , which has already been discussed. Combining the results of [37] and [40] we see that the network in equation (4) can be specified such that all solutions must approach the unique periodic solution.

In reference [41] , Duffin considers networks consisting of transformers and resistors, arbitrarily interconnected to a set of generators. The network equations are first integrated with respect to time. The integrated equations then are similar in form to the equations analyzed in [37] and [40]. In these equations the permeability of the core plays the same role as the resistance in the previous equations. Thus, Duffin again uses Liapunov - like arguments to analyze these new equations.

## Example 29, [42]    Jone's 2nd Order Equation

Jones considers a class of nonlinear second order differential equations which has occurred in astrophysics, atomic physics and mechanics. Jones presents a set of sufficient conditions which guarantees that no solution of the system has an arbitrarily large positive zero. In the proof of his main theorem, Jones uses an "amplitude variable" which is really a <u>Liapunov function</u>.(For this reason this example is included in the report.)

The system is defined by the following equation:

$$y'' + \sum_{i=1}^{n} f_i(x) \, y^{2i-1} = 0. \tag{1}$$

All of the coefficients of (1) are assumed to be real-valued, bounded, and Lebesgue - measurable functions of x, x $\geqslant$ 0. It can be proved that a solution of (1) will be an absolutely continuous, real-valued function with an absolutely continuous derivative satisfying the differential equation almost everywhere in the sense of Carathéodory.

## Theorem, [42]

H)  If  (i)  $f_n(x)$ has a positive lower bound for x $\geqslant$ 0,

   (ii)  $f_i(x) \geqslant 0$, $f'_i(x) \leqslant 0$ for i = 1, ..., n,

   (iii)  $f'_i(x)$ are Lebesgue - measurable,

   (iv)  n is a positive integer greater than 1,

   (v)  $\displaystyle\int_{0}^{\infty} \left\{ \sum_{i=1}^{n} f_i(x) \, x^{2i-1} \right\} dx < \infty$ ,

C)  Then there is no solution, y(x), of (1) with arbitrarily large positive zeros.

## Outline of the Proof

Jones defines an "amplitude variable" in the following way:

$$R(x) = \frac{1}{2} (y')^2 + \sum_{i=1}^{n} f_i(x) y^{2i}, \tag{2}$$

where $y(x)$ is a solution of (1). This amplitude variable is really a Liapunov function; $R(x)$ is positive for $x > 0$ and

$$R'(x) = \sum_{i=1}^{n} f_i'(x) y^{2i} \leq 0. \tag{3}$$

From (2) and (3) we conclude that for any solution of (1), $y'(x)$ remains bounded as $x \longrightarrow \infty$. The author then constructs a contradiction proof to verify that the conclusion is valid. (The main vehicle in this proof was a Liapunov function, $R(x)$).

## Example 30, [43]    Utz's 2nd Order Equation

In reference [43], Utz considers various sets of sufficient conditions which guarantee boundedness or asymptotic stability of solutions of 2nd order systems. In theorem 5, Utz proves his results through the use of a Liapunov function. We will now outline Utz's contribution as reported in [43].

The systems which are investigated have coefficients which are differentiable, and are such that $x(t) \equiv 0$ is a solution of each system.

## Definition 1

A solution, $x(t)$, of the system is called oscillatory if it has positive maxima and negative minima for arbitrarily large t.

In the first two theorems, the system being discussed is defined by:

$$\ddot{x} + f(x,\dot{x}) + g(x) = 0. \tag{1}$$

## Theorem 1

(H) If (i) $f(x, \dot{x}) \geqslant 0$ for $x, \dot{x}$,

(ii) $x\ g(x) > 0$ for all $x \neq 0$,

(iii) $\displaystyle\int_0^x g(x)\ dx \longrightarrow \infty$ as $x \longrightarrow \infty$,

(iv) $x$ is a nonzero solution of (1), valid for all $t > 0$,

(C) then $x$ is bounded and oscillatory as $t \longrightarrow \infty$, or $x$ monotonically approaches 0 as $t \longrightarrow \infty$.

## Theorem 2

(H) If (i) $f(x, \dot{x}) > 0$, except at a discrete number of points,

(ii) $g(x)$ is an odd function,

(iii) $x$ is an oscillatory solution of (1),

(C) then the amplitudes of the oscillations of $x$ are monotonically decreasing.

In the next three theorems, the following equation is considered:

$$\ddot{x} + f(x)\ \dot{x} + g(x) = 0 , \qquad (2)$$

where $f'(x) = df/dx$.

## Theorem 3

(H) If (i) $x\ g(x) > 0$ for all $x \neq 0$,

(ii) $g(x)/x \longrightarrow \infty$ as $x \longrightarrow \infty$,

(iii) there exist constants $b, B > 0$ such that for all real $x$,

$\left| f(x) - b\ g(x) \right| \leq B\,|x|$,

(iv) $x$ is a nonzero solution of (2) for $t > 0$,

(C) then $x$ is bounded and oscillatory, or $x$ monotonically approaches zero as $t \longrightarrow \infty$.

## Theorem 4

(H) If (i) there exists positive constants a and b such that $b > a^2$ and

$$b + 4a^2 \geqslant a \, F(x) \geqslant G(x) \geqslant b > 0 ,$$

where $F(x) = f(x)/x$ and $G(x) = g(x)/x$,

(ii) x is any nonzero solution of (2) valid for all large t,

(C) then x is bounded and oscillatory, or x monotonically approaches zero as

$t \longrightarrow \infty$.

## Theorem 5

(H) If (i) $xf(x) > 0$, $xg(x) > 0$ for all $x \neq 0$,

(ii) $g(x)/x \longrightarrow \infty$ as $x \longrightarrow \infty$,

(iii) x is any nonzero solution of (2) valid for all large t ,

(C) then x is oscillatory, or x monotonically tends to zero as $t \longrightarrow \infty$.

## Notes about the Theorems

(1) The above theorems are independent of each other as we can show by example.

In equation (1) let $f(x, \dot{x}) = 1$ and $g(x) = \left\{ \dfrac{x}{x^2 + 1} \right\}$. Then Theorem 1 is valid, but Theorems 3 and 4 do not apply. In equation (2), if $f(x) = g(x) = 9 x (\exp \left[-x^2\right] + x^2)$ and $b = B = 1$, then Theorem 3 applies but not Theorems 1 and 4. In equation (2), if $f(x) = 3x (\exp \left[-x^2\right] + 1)$ , $g(x) = 3x$, and $b = 3$ and $a = 3/2$, then Theorem 4 applies but not Theorems 1 and 3. In equation (2), if $g(x) = x^3$ and $f(x) = 3x (\exp \left[-x^2\right] + 1)$, then Theorem 5 applies, but not Theorems 1, 3, and 4.

(2) In the proofs of Theorems 1 through 4, non-Liapunov methods were employed; but in the proof of Theorem 5, a Liapunov function was used.

The system in equation (2) was rewritten in the following form:

$$\dot{x} = y - f(x)$$

$$\dot{y} = - g(x) .$$

(3)

The choice for a Liapunov function was

$$V(t) = 2 \int_0^t g(x) \dot{x} \, d t + y^2 .$$

By the hypotheses of Theorem 5, $V(t)$ is positive definite. The time derivative of $V(t)$ with respect to (3) is

$$\dot{V}(t) = - 2g(x) f(x) \leq 0 .$$

Also, from Theorem 5, $V \longrightarrow \infty$ as $t \longrightarrow \infty$. Therefore, the conclusion of Theorem 5 follows from Liapunov theory.

## Example 31, [44, 45, 46, 47]    Volterra's Equation

In references [44] to [47] , the integro-differential equation of Volterra is considered. This equation occurs in the study of reactor dynamics, nonlinear oscillators with hereditary terms, and in many other physical applications. The scalar form of the equation is given by

$$\dot{x}(t) = - \int_0^t a(t - \tau) \, g \, (x(\tau)) \, d\tau \quad ( \quad ) \quad .$$

(1)

In reference [44], a theorem dealing with the asymptotic stability of the null solution of (1) is presented. In reference [47] , the existence and uniqueness of the solutions of (1) are considered. In reference [45] , the solutions of (1) are investigated as $t \longrightarrow \infty$ for the case where $a(t)$ is completely monotonic over the interval $[0, \infty)$ and where $g(x)$ is thought of as a "nonlinear spring" term. The results in this reference, [45] , are weaker than the results in [44]; but the theorem proved in [45] brings "under one roof" several different notions

of positivity: such as, Liapunov functions, completely monotonic functions, and positive type kernel functions.

In reference [46] , Kemp investigates the same nonlinear integro-differential equation as given in (1), but for the n-dimensional case. The hypotheses in Theorems 1 and 2, which are presented below, are directly generalized to higher order and the corresponding Liapunov functions, V(t) and E(t) as given below, are also directly generalized. The generalization of the results for (1) involves some labor, but no new concepts.

We will now present the two theorems, with notes about their proofs, which occur in [44] and [45].

## Theorem 1, [44]

(H)  If  (i) a(t) is continuous over $[0, \infty)$ ,

(ii) $(-1)^K a^{(K)}(t) \geqslant 0$ over $(0, \infty)$, where $K = 1, 2, 3$ and $a^{(K)}$ is the K-th derivative,

(iii) g(x) is continuous in $(-\infty, \infty)$,

(iv) $xg(x) > 0$, $x \neq 0$,

(v) $G(x) = \int_0^x g(N) \, dN \longrightarrow \infty$ as $|x| \longrightarrow \infty$ ,

(vi) $a(t) \not\equiv a_0$ (thus, no periodic solutions exist),

(vi) U(t) is any solution of (1) over $[0,\infty)$,

(C)  then,

$$\lim_{t \longrightarrow \infty} U^{(j)}(t) = 0, \quad j = 0, 1, 2. \tag{2}$$

## Outline of the Proof

First, the author in [44], states and proves several lemmas dealing with the boundedness of $\dot{a}$, $t\,\ddot{a}$, and $t^2\,\dddot{a}$. Next, the following energy or Liapunov function is specified:

$$E(t) = G(u(t)) + \frac{a(t)}{2} \left[ \int_0^t g(u(\tau))\, d\tau \right]^2 +$$

$$- \frac{1}{2} \int_0^t \dot{a}\,(t-\tau) \left[ \int_\tau^t g(u(s))\, ds \right]^2 d\tau \;\geqslant\; 0 . \tag{3}$$

By integration by parts, the time derivative of $E(t)$ relative to (1) yields

$$\dot{E}(t) = \frac{\dot{a}(t)}{2} \left[ \int_0^t g\,(u(\tau))\, d\tau \right]^2 - \frac{1}{2} \int_0^t \ddot{a}\,(t-\tau) \left[ \int_\tau^t g(u(s))\, ds \right]^2 d\tau ,$$
$$\leq 0 . \tag{4}$$

From (3) and (4), we conclude that

$$G(u(t)) \leq E(t) \leq E(0) = G\,(u_0) ,$$

where $u_0 = u(0)$. By hypothesis, it follows that

$$\left| u(t) \right| \leq K\,(u_0) < \infty \quad \text{for} \quad [0, \infty) . \tag{5}$$

Finally, it can be shown that as $u_0 \longrightarrow 0$, $K(u_0) \longrightarrow 0$; and also, it can be shown that $\dot{u}$ and $\ddot{u} \longrightarrow 0$ as $t \longrightarrow \infty$. (The crux of this proof was the use of a Liapunov function as described in (3) and (4).)

## Theorem 2, [45]

(H)  If   (i) $a(t)$ is continuous in $[0, \infty)$

  (ii) $(-1)^K a^{(K)}(t) \geqslant 0$ for $K = 0, 1, 2, \ldots$ and in $(0, \infty)$,

  (iii) $g(x)$ is continuous in $(-\infty, \infty)$,

  (iv) $xg(x) > 0$  $(x \neq 0)$,

(v) $\quad G(x) = \int_0^x g(N)\, dN \longrightarrow \infty$ as $|x| \longrightarrow \infty$,

(vi) $\quad a(t) \not\equiv a_0$,

(vii) $\quad u(t)$ is any solution of (1) in $[0, \infty)$,

(C) then

$$\lim_{t \longrightarrow \infty} u^{(j)}(t) = 0, \quad (j = 0, 1, 2). \tag{6}$$

## Notes about the Theorems

(1)  In Theorem 1, only $K = 0, 1, 2, 3$ in (ii) was required of $a(t)$, not the complete monotonicity as required here.

(2)  The Liapunov function used in the proof of this theorem is given by

$$V(t) = G(u(t)) + \frac{1}{2} \int_0^t \int_0^t a(\tau + s)\, g(u(t-\tau))\, g(u(t-s))\, d\tau ds, \tag{7}$$

Where $V \geqslant 0$ if the second term is nonnegative.  The physical interpretation of $V(t)$ is that the first term is the potential energy of the system and the second term is the kinetic energy.  The kinetic energy term will be nonnegative if $a(\tau + s)$ is a kernel of the <u>positive type</u> [48, p 270] on the square $0 < \tau, s < t$, for each $t$ in $(0, \infty)$.  Differentiating (7) with reference to (1), yields

$$\dot{V}(t) = \int_0^t \int_0^t \dot{a}(\tau + s)\, g(u(t-\tau))\, g(u(t-s))\, d\tau\, ds. \tag{8}$$

If $- \dot{a}(\tau + s)$ is a kernel of the <u>positive type</u> on $0 < \tau, s < t$ for each $t$ in $(0, \infty)$, then $\dot{V}$ will be nonpositive.

(3)  In [48, p 160], it is proved that $a(t)$ satisfies the required conditions in <u>Item (2)</u> if the following is satisfied:

$$a(t) = \int_0^\infty \exp\left\{ - \xi\, t \right\}\, d\alpha(\xi), \tag{9}$$

where $\alpha(\infty) < \infty$ and $\alpha$ is nondecreasing on $0 < \xi < \infty$.

(4)  $E(t) \equiv V(t)$ if $a(t) \equiv a_0$ for all t.

## Outline of the Proof of Theorem 2

We assume that $V(t)$ and $a(t)$ are defined as in (7) and (9). We then define $\Gamma(\xi, t)$ by

$$\Gamma(\xi, t) = \int_0^t \exp\left\{-\xi(t - \tau)\right\} g(u(\tau)) d\tau,$$  (10)

where $(0 \leq \xi, t < \infty)$. Thus, we can write V and $\dot{V}$ in terms of $\Gamma$:

$$V(t) = G(u(t)) + \frac{1}{2} \int_0^\infty \Gamma^2(\xi, t) d\alpha(\xi) \geq 0,$$  (11)

for $0 \leq t < \infty$, and

$$\dot{V}(t) = -\int_0^\infty \xi \Gamma^2(\xi, t) d\alpha(\xi) \leq 0,$$  (12)

for $0 \leq t < \infty$. Equations (11) and (12) reduce to the inequality:

$$G(u(t)) \leq V(t) \leq V(0) = G(U_0),$$  (13)

for $(0 \leq t < \infty)$. From the hypotheses and (13), we have $|U(t)| \leq K < \infty$,

where $K \longrightarrow 0$ as $U_0 \longrightarrow 0$. Similar results are obtained for $\dot{U}$ and $\ddot{U}$. Thus,

the conclusions of the theorem are valid.

## Example 32, [49]        Exponential Stability

In this "example" we will outline the paper of <u>Bhatia</u>, giving the important

definitions, theorems and examples.

The following linear, n-th order differential system is to be examined:

$$\dot{x} = A(t) x.$$  (1)

Also, the nonlinear system given below is analyzed:

$$\dot{\underline{x}} = \underline{A}(t) \underline{x} + \underline{f}(t, \underline{x}), \quad \underline{f}(t, \underline{0}) = \underline{0},$$

for $t \geqslant 0$. The elements of $\underline{A}$, $a_{ij}(t)$, are defined and continuous on $[0, \infty)$.

## Definition 1

"The solution $\underline{x} = \underline{0}$ of (1) is said to be <u>exponentially stable</u>, if there exist positive constants $\alpha$ and $a$ such that for any solution $\underline{x}(t)$ of (1), $\underline{x}(t_o) = \underline{x}_o$, the inequality

$$\| \underline{x}(t) \| \leq \alpha \| \underline{x}_o \| \exp \left\{ - a(t - t_o) \right\},$$

holds for $t \geqslant t_o$."

Let $\underline{B}(t)$ be a symmetric matrix defined and continuous on $[0, \infty)$.

## Definition 2

"The quadratic form $\underline{x}_T \underline{B}(t) \underline{x}$ is <u>positive definite</u> if there exists a positive constant $b$ such that

$$\underline{x}_T \underline{B}(t) \underline{x} \geqslant b \, \underline{x}_T \underline{x}, \quad t \geqslant 0."$$

## Definition 3

"The quadratic form $\underline{x}_T \underline{B}(t) \underline{x}$ has <u>property P</u> if it is positive definite and if it is positive definite and if the elements, $b_{ij}(t)$, of $\underline{B}(t)$ are uniformly bounded on $[0, \infty)$."

Bhatia has proved that the necessary and sufficient conditions for $\underline{B}(t)$ to have property P is that there exists constants $b_1$ and $b_2$ such that

$$b_1 \, \underline{x}_T \underline{x} \leq \underline{x}_T \underline{B}(t) \underline{x} \leq b_2 \, \underline{x}_T \underline{x}, \quad t \geqslant 0.$$

If $V = \underline{x}_T \underline{B}(t) \underline{x}$, where the elements of $\underline{B}$ have continuous derivatives on $[0, \infty)$, then the time derivative of $V$ with respect to equation (1) is given by

$$\dot{V}_1 = \underline{x}_T \left\{ \dot{\underline{B}} + \underline{A}_T \underline{B} + \underline{B} \, \underline{A} \right\} \underline{x}.$$

Theorem 1, [50]    (Malkin)

(H) If   (i)   $\underline{x}$ = $\underline{0}$, of (1), is exponentially stable,

   (ii)   the elements of $\underline{A}$ in (1) are uniformly bounded on $[0, \infty)$ ,

   (iii)   $\overline{\underline{X}}$ (t) is the fundamental matrix solution of (1),

   (iv)   $\underline{x}_T \underline{C}(t)$ $\underline{x}$ possesses property $\underline{P}$,

(C)   then for each $\underline{C}$ there exists a $\underline{B}$ such that $\underline{x}_T \underline{B}(t)$ $\underline{x}$ possesses property $\underline{P}$ and

$$V = \underline{x}_T \underline{B}(t) \underline{x} = \int_t^{\infty} \left\{ \overline{\underline{X}}(\tau) \overline{\underline{X}}^{-1}(t) \underline{x} \right\}_T \underline{C}(\tau) \left\{ \overline{\underline{X}}(\tau) \overline{\underline{X}}^{-1}(t) \underline{x} \right\} d\tau, \qquad (7)$$

where $\dot{V}_1 = - \underline{x}_T \underline{C}(t) \underline{x}$ .

   Roseau improved Malkin's result in the following theorem.

Theorem 2, [51, 52]    (Roseau)

(H)   If   (i)   $\underline{x}$ = $\underline{0}$ of (1) is exponentially stable,

   (i i)   matrix $\underline{A}$ (t ) satisfies the condition

$$\underline{R}(S, t) = \int_t^s A(\tau) \left\{ \overline{\underline{X}}_T^{-1}(t) \underline{X}(\tau)_T \right\}_T d\tau \to 0 \text{ as } (s - t) \to 0, \qquad (8)$$

uniformly on s $\geqslant$ t $\geqslant$ o,

   (iii)   $\underline{x}_T \underline{C}(t)$ $\underline{x}$ has property $\underline{P}$,

(C) then there exists a quadratic form

   $V = \underline{x}_T \underline{B}(t) \underline{x}$ having property P and satisfying equation (7), where

$\dot{V}_1 = - \underline{x}_T C(t) \underline{x}$.

Example 1

   Consider the scalar equation

$$\dot{r} = (2t \cos t^2 - 1) r,$$

where $r(t_0) = r_0$. The general solution is

$$r(t) = r_0 \exp \left\{ \sin t^2 - \sin t_0^2 - t + t_0 \right\}.$$

thus, we have

$$|r(t)| \leq |r_0| e^2 \exp\left\{-(t - t_0)\right\} , t \geq t_0,$$

which proves that we have exponential stability. Malkin's formula, (7), gives the Liapunov function

$$V = r^2 \int_t^\infty \exp\left\{2 (\sin \tau^2 - \sin t^2) - 2 (\tau - t)\right\} d\tau,$$

where $\dot{V}_1 = -r^2$ and $V$ satisfies

$$r^2 e^{-4}/2 \leq V \leq r^2 e^4/2 .$$

Thus, both $V$ and $\dot{V}_1$ have property $\underline{P}$. Notice however that $(2t \cos t^2 - 1)$ is neither bounded nor does Roseou's condition hold. That is,

$$R(s, t) = \int_t^s (2\tau \cos \tau^2 - 1) \exp\left\{\sin \tau^2 - \sin t^2 - (\tau - t)\right\} d\tau$$

$$= \exp\left\{\sin s^2 - \sin t^2 - (s - t)\right\} - 1,$$

where if $s = t + 1/t$, $s - t \longrightarrow 0$ as $t \longrightarrow \infty$ but $R(t + 1/t, t) \longrightarrow\!\!\!\!\!/ \ 0$. However, the following relationship does hold:

$$e^{-2} |r_0| \exp\left\{-(t - t_0)\right\} \leq |r(t)| \leq e^2 |r_0| \exp\left\{-(t - t_0)\right\},$$

$t \geq t_0$. This example points out the need for a different condition which must be placed on the solutions of equation (1). Bhatia calls this condition "exponential decay."

### Definition 4

"The solutions of (1) are said to <u>decay exponentially</u> if there exists positive constants $a, \alpha, b, \beta$ such that every solution $\underline{x}(t)$, $\underline{x}(t_0) = \underline{x}_0$, of (1) satisfies the inequalities

$$\|\underline{x}_0\| \beta \exp\left\{-b(t - t_0)\right\} \leq \|\underline{x}\| \leq \|\underline{x}_0\| \alpha \exp\left\{-a(t - t_0)\right\} \tag{9}$$

for $t \geq t_0$."

In the following five theorems the theory of exponential decay, as applied to equation (1), is summarized.

## Theorem 3, [49]

"The solutions of (1) decay exponentially if and only if there exists a quadratic form

$V = \underline{x}_T \underline{B}(t) \, \underline{x}$ such that $V$ and $\dot{V}_1$ both have property $\underline{P}$."

## Theorem 4, [49]

"The solutions of (1) decay exponentially if and only if there exists a positive definite form $V$ of order $m$ with uniformly bounded coefficients such that $-\dot{V}_1$ is positive definite and has uniformly bounded coefficients."

## Theorem 5, [49]

"If the solutions of (1) decay exponentially, then

$$k \leq \int_t^{\infty} \left\{ \exp \left( 2 \int_t^{\tau} Tr\{\underline{A}(s)\} \, ds \right) \right\} d\tau \leq K, \tag{10}$$

for $t \geq 0$ and for positive constants k, K."

## Theorem 6, [49]

"If the solution $\underline{x} = \underline{0}$ of (1) is exponentially stable and if there is a positive constant K such that

$$\int_t^{\infty} \exp \left\{ 2 \int_t^{\tau} Tr\{\underline{A}(s)\} \, ds \right\} d\tau \geq K, \tag{11}$$

for $t \geq 0$, then the solutions of (1) decay exponentially."

Theorem 7, [49]

"A necessary and sufficient condition for the existence of a quadratic form V such that V and $\dot{V}_1$ both have property $\underline{P}$ is that the solution $\underline{x} = \underline{0}$ of (1) be exponentially stable and the condition (11) holds".

Bhatia now discusses a more general concept, which he calls "generalized exponential decay" (g.e.d.).

Definition 5

"The solutions of (1) are said to exhibit g.e.d. if there exists a nondecreasing function $\phi(t)$ possessing a continuous derivative such that $\phi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, and if there exist four positive constants $\alpha$, a, $\beta$, b such that every solution $\underline{x}(t)$ of (1) satisfies the inequalities

$$\|\underline{x}_0\| \; \beta \; \exp\left\{ - b \left[ \phi(t) - \phi(t_0) \right] \right\} \leq \|x(t)\| \leq$$

$$\leq \|\underline{x}_0\| \; \alpha \; \exp\left\{ -a \left[ \phi(t) - \phi(t_0) \right] \right\} , \tag{1}$$

for $t \geqslant t_0$."

Example 2

Consider the scalar equation

$$\dot{r} = -\left\{ \frac{1}{1 + t} \right\} \; r ,$$

whose general solution is

$$r(t) = r_0 \; \exp\left\{ - (\log(t + 1) - \log(t_0 + 1)) \right\} .$$

We have g.e.d. with $\alpha = a = \beta = b = 1$ and $\phi(t) = \log(t + 1)$. But we do not have exponential stability.

The following theorem gives a necessary and sufficient condition for g.e.d.

## Theorem 8, [49]

"The solutions of (1) exhibit g.e.d. if and only if there exist two quadratic

forms $V = \underline{x}_T \underline{B}(t) \underline{x}$ and $W = \underline{x}_T \underline{C}(t) \underline{x}$

having property $\underline{P}$ and a nonnegative continuous function $\Theta(t)$ such that

$$\int_t^\infty \Theta(\tau) \, d\tau = + \infty \text{ and } \dot{V}_1 = - \Theta(t) W."$$

## Example 3

Consider the system

$$\dot{x} = y, \quad \dot{y} = -x - \frac{2}{t} y.$$

If we choose

$$V = x^2 + y^2 + \frac{2}{t} x y ,$$

then V has property $\underline{P}$ for $t \geqslant 2$, and

$$\dot{V}_1 = - \frac{2}{t} (x^2 + y^2 + \frac{3}{t} xy) .$$

Thus, let $W = (x^2 + y^2 + \frac{3}{t} x y)$ and $\Theta(t) = 2/t$. Notice that W has property

$\underline{P}$ for $t \geqslant 2$, and $\int_t^\infty \frac{2}{t} dt = + \infty$. Therefore the solutions of the system exhibit g.e.d.

## Note

The g.e.d. implies uniform stability, but it does not in general imply uniform asymptotic stability since in linear systems uniform asymptotic stability is equivalent to exponential stability.

The next theorem is concerned with the exponential stability of the solutions of (1).

Theorem 9, [49]

"The solution $\underline{x} = \underline{0}$ of (1) is exponentially stable if and only if there exists a continuous function $v(t, \underline{x})$ having the properties:

(i) $v(t, \underline{0}) = 0$ and there are positive constants a and b such that

$$a \, \|\underline{x}\| \le v(t, \underline{x}) \le b \, \|\underline{x}\| \quad , t \ge 0,$$

(ii) $v(t, \underline{x})$ is locally lipschitzian in $\underline{x}$,

(iii) and, where C is a positive constant, we have

$$v^*_1 = \lim_{h \to 0+} \sup \frac{1}{h} \left\{ v(t + h, \underline{x} + h \, \underline{A}(t) \, \underline{x}) - v(t, \underline{x}) \right\}$$

$$\le - C \, \|\underline{x}\| \quad ."$$

Note

The result in Theorem 9 can be extended to cover the nonlinear system

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}) , \quad \underline{f}(t, \underline{0}) = \underline{0},$$

if (i) and (ii) hold for some region $\|\underline{x}\| < R$, R a positive constant, and if in $v_1^*$, $\underline{A}(t) \, \underline{x}$ is replaced by $\underline{f}(t, \underline{x})$.

The final theorem is a slight generalization of a theorem due to Perron.

Theorem 10, [49]

"Suppose the origin $\underline{x} = \underline{0}$ of (1) is exponentially stable and the function $\underline{f}(t, \underline{x})$ in (2) is continuous and satisfies the condition $\underline{f}(t, \underline{x}) = 0 \, ( \, \|\underline{x}\| \, )$. Then the origin of system (2) is exponentially stable." (Bhatia does not require boundedness of the elements of $\underline{A}(t)$.)

### RANDOM CONTRIBUTIONS TO STABILITY THEORY

(1) In reference [53], Rosenbrock considers the nonlinear, nonautonomous system defined by

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t), \, \underline{f}(\underline{0}, t) = \underline{0}.$$

In analyzing this system, the above equation was replaced by $\dot{\underline{x}} = \underline{A}(\underline{x}, t)\,\underline{x}$, and conditions on the elements of $\underline{A}$ were determined to ensure stability. In reference [54], the method used in [53] is applied to the n-th order differential equation

$$x^{(n)} = f(x, \dot{x}, \dots, x^{(n-1)}, t).$$

This equation is best analyzed if it is replaced by

$$x^{(n)} + a_n x^{(n-1)} + \dots + a_2 \dot{x} + a_1 x = 0,$$

$$a_i = a_i(x, \dot{x}, \dots, x^{(n-1)}, t).$$

Rosenbrock states that if $\lambda_1, \dots, \lambda_n$ are the roots of the equation

$$\lambda^n + a_n \lambda^{n-1} + \dots + a_2 \lambda + a_1 = 0,$$

then knowledge of the $\lambda_i(x, \dot{x}, \dots, x^{(n-1)}, t)$ is equivalent to knowledge of the $a_i$. Consequently conditions on the $a_i$ which ensure stability can be replaced by conditions on $\lambda_i$; and this is what Rosenbrock does. The conditions which are given guarantee that $\underline{x} = \underline{0}$ is uniformly asymptotically stable. Some of the theorems proved by Rosenbrock make use of Liapunov theory.

In reference [55], Rosenbrock studies the stability properties of the second order system:

$$\ddot{x} + a_2(x, \dot{x}, t)\,\dot{x} + a_1(x, \dot{x}, t)\,x = 0.$$

He obtains a slightly stronger result for this system in [54].

(2) In reference [56], Brayton & Moser use Liapunov theory in their derivation of stability criteria for nonlinear electrical networks.

(3) In reference [57], Hochstadt considers the second order system

$$\ddot{y} + P(t)\,\dot{y} \pm q^2 y = 0,$$

where $\underline{P}(t)$ is bounded and non-negative, and q is a real positive constant. When the "minus sign" is used, he proves that unbounded solutions exist. When the

"plus sign" is used, he proves that only bounded solutions exist.

(4)  In reference [58], <u>Bebernes and Vinh</u> consider the linear time-varying system given by:

$$\dot{\underline{x}} = \left\{ \underline{F}(t) + \underline{G}(t) \right\} \underline{x} \, ,$$

where $\underline{x}$ is an n-dimensional vector, and $\underline{F}$ and $\underline{G}$ are n x n-matrices defined on $[0, \infty)$. The main result of the paper is as follows:

"if $\bar{\underline{X}}(t)$ is the fundamental matrix solution of $\dot{\underline{x}} = \underline{F}(t) \underline{x}$, $\underline{G}(t)$ is continuous in $[0, \infty)$, and if

$$\int_0^\infty \left\| \bar{\underline{X}}^{-1}(s) \ \underline{G}(s) \ \bar{\underline{X}}(s) \right\| \ ds < \infty \, ,$$

then every solution $\underline{Y}(t)$ of the original system can be expressed as $\underline{Y}(t) = \bar{\underline{X}}(t) \underline{c}(t)$, where $\underline{c}$ is defined by

$$\dot{\underline{c}} = \bar{\underline{X}}^{-1} \ \underline{G} \ \bar{\underline{X}} \ \underline{c} \, ,$$

bounded in $[0, \infty)$, and the $\lim\limits_{t \to \infty} \underline{c}(t)$ exists and is unique."

(5)  In reference [59], <u>Struble</u> studies the system defined by:

$$\dot{\underline{x}} = \underline{A} \ \underline{x} + \epsilon \ \underline{f} \ (\underline{x}, t, \epsilon) \, ,$$

where $\underline{x}$ is an n-vector, $\underline{A}$ is a constant matrix and $\epsilon$ is a scalar parameter. <u>Struble</u> gives a more detailed picture of the approach of a solution of this system to its equilibrium solution than that afforded in the usual stability theorems.

(6)  In reference [60], <u>LaSalle and Wonham</u> give a summary of the stability papers given at the 2<u>nd</u> International Conference on Automatic Control Theory.  We give some of their comments in the following discussion.

(a)  <u>Paper No. 103</u> - "On the Estimation of the Decaying time", <u>H. Ling</u>(Communist China) the paper deals with Liapunov - like stability theory for compact manifolds.  The work is not necessarily new.

(b) <u>Paper No. 415</u> - "Eventual Stability", <u>J. P. LaSalle and R. J. Roth</u> (U.S.A.).

A Liapunov-like theory has been developed in this paper for the new concept of

"eventual stabilities" and this theory can be applied to certain types of problems

when Liapunov theory is not applicable. The paper also contains a theorem on

the asymptotic stability of noncompact manifolds.

(c) <u>Paper No. 324</u> "Nonlinear Stability Analysis for Stricted Nonlinearities Using

the Second Method of Liapunov" - <u>H. Nour Eldin</u>, the author considers nonlinear

control systems and uses Liapunov's second method for stability analysis. The systems

are Lure type containing a single nonlinearity. The paper contains serious errors and

may <u>not</u> be too useful.

(d) <u>Paper No. 420</u> "The Use of the technique of Linear Bounds for Applying the Direct

Method of Liapunov to a Class of Non-Linear and time - Varying Systems" - <u>R. A. Nesbit (U.S.A</u>

the author shows how Liapunov's second method may be used to obtain estimates of a class

of functions for which a given controller will operate satisfactorily. This problem

arises when there is uncertainty as to the exact mathematical description of the forces

and dynamics of the system being controlled.

(7) In reference [61] , <u>Wong</u> gives two boundedness theorems for the second order system

defined by

$$\ddot{u} + a(t) f(u) g(\dot{u}) = 0,$$

where $f(u)$, is integrable and $uf(u) > 0$. The function $g(u)$ is positive continuous.

(8) In reference [62] , <u>Zubov</u> considers the system

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y),$$

where the $f_i$ are given in a region G of the x y - plane. The $f_i$ are real, continuous

and twice differentiable. <u>Zubov</u> assumes that the system has periodic solutions. He

proves many theorems dealing with the Liapunov stability and instability of the

periodic solutions of the system.

(9) In reference [63], <u>Matrosov</u> considers the general case of nonsteady motion. In the obtained criteria two Liapunov functions are used. For the case of nonuniform asymptotic stability, the requirement of an infinitely small higher limit is removed, which leads to the modification of theorems of Krasovskii, Zubov, and Reisig. The application of the method to a nonstationary gyroscopic system with dissipation is discussed.

(10) In reference [64], <u>Matrosov</u> studies nonlinear, nonautonomous systems through the use of several "Liapunov functions". In this connection each V-function can satisfy less rigid requirements than the one function occurring in the corresponding theorems of Liapunov's second method. The work is based on Chaplygin's theory of differential inequalities. The stability theorems obtained with the use of several V-functions enables the author to construct tests for stability and instability which utilize the properties of derivatives of the V - functions of higher order than the first. <u>Matroso</u> considers tests with derivatives of first and second order.

<u>Matrosov</u> applies this theory to the problem of the stability in the sense of Liapunov of bodies with variable moss, and to a second order nonlinear, nonautonomous system of the form:

$$\dot{x}_1 = (\sin t + e^{-t}) x_1 + (\sin t - e^{-t}) x_2 - \sin^2 t (x_1^3 + x_1 x_2^2)$$

$$\dot{x}_2 = (\sin t - e^{-t}) x_1 + (\sin t + e^{-t}) x_2 - \sin^2 t (x_1^2 x_2 + x_2^3).$$

(11) In reference [65], <u>Chzhan Sy-in</u> considers problems on the stability of motion over a finite interval of time. He considers the stability of the motion of the following system during the finite interval $[t_o, T]$ :

$$\dot{\underline{x}} = \underline{P}(t) \underline{x} ,$$

where $\underline{P}$ is a real, bounded, continuous matrix function of time . Liapunov functions are used in all of the stability studies. Also, he considered systems with slowly

changing coefficients; that is, $\underline{P}(t) = \underline{C} + \epsilon \underline{f}(t)$, where $\underline{C}$ is a constant and $\epsilon$ is sufficiently small.

Chzhan Sy-In considers systems with continuously acting disturbances and nonlinear systems. He gives particular examples of the various systems.

Other references on finite time stability are listed in the reference list, [66] to [69].

(12) In [70], Zubov considered the stability of the null solution in doubtful cases. He gave several definitions of stability and used a Liapunov approach in his analysis.

(13) In [71], Razumikhin considered a linear time-varying system $\dot{\underline{x}} = \underline{A}(t) \underline{x}$, where $\underline{A}$ is a continuous, bounded function of time. The Liapunov function which he used was of the form, $V = \underline{x}_T \underline{B} \underline{x}$, where $\underline{B}$ is constant. He also determined the region of state space in which V is a Liapunov function of the above system. And, he extended his work to nonlinear systems.

(14) In [72], Aizerman and Gantmakher studied the stability of periodic solutions of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ by applying non-Liapunov methods -- they used the variational equations corresponding to the above system.

(15) In [74], Livartorskii dealt with the stability of any solution of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, where $\underline{f}$ is a discontinuous non-periodic function. New criteria for stability was introduced. Liapunov functions were used to prove the stability theorems.

(16) In [75], Kalinin investigated periodic motions in the case of two zero roots by using Liapunov methods.

(17) In [76], Chetaev generalized the theorem of Poincare' and Liapunov to the general case of stable motions of conservative systems. He used a Liapunov function given by a Hermitian form.

(18) In [77] to [81], the various authors studied the stability of the equilibrium positions for discontinuous systems through the use of Liapunov methods.

(19) In [82] , <u>Kuz'min</u> considered the stability of mechanical systems by employing quadratic forms as Liapunov functions.

(20) In [83] , <u>Markhashov</u> studied the critical cases of stability of stationary motions (according to Liapunov) by employing certain simple facts from the theory of continuous groups of transformations.

(21) In [84] , <u>Klimushchev and Krasovskii</u> proved that under the assumption of uniform asymptotic stability of the degenerate first approximation system and the asymptotic stability of a certain auxiliary system, a certain class of systems of differential equations with a small parameter among its derivative terms was asymptotically stable. The method of Liapunov was employed.

(22) In [85] , <u>Regish</u> discussed nonuniform stability of time-varying systems from a function - space approach.

(23) In [86] , <u>Krein</u> studied certain problems dealing with characteristic values and the Liapunov zones of stability.

(24) In [87] , <u>Hale and Stokes</u> considered the asymptotic stability of nonautonomous systems of linear and nonlinear differential equations. The approach used was to consider the integral representation of these systems and then apply certain differential inequalities.

(25) In [88] and [89] , the authors applied Liapunov's second method to certain physical applications:

linear and nonlinear stability problems in plasma physics and stability problems in adaptive control systems, respectively.

(26) In [90] , <u>Tomovic'</u> discussed many aspects of stability and sensitivity analysis. They were:

control systems, types of disturbances, orbital stability, structual stability (not the topological variety), conditionally stable systems, stability in finite

time, and Liapunov's second method.

(27) In [92] , _Jaffe_ applied Liapunov's second method to the optimal control problem.
In [93] , _Vogt_ defined relative stability and then studied this type of stability
in differential systems through the use of Liapunov functions. In [94] , _Seifert_
studied orbital stability and gave several examples to illustrate his work.

(28) In the references, [95] to [129] , we list some of the contributions to bounded-
ness and stability of nonlinear systems given by the _Italian Mathematicians_ during the
years 1951-1961. In [95] and [96], the asymptotic properties of the solutions of
$\ddot{x} + \phi(\dot{x}) + x = 0$ were studied. In [97, 98, 99] , the bounds and asymptotic
stability of the solutions of $\ddot{x} + f(x)\dot{x} + g(x) = 0$ were discussed. In [100] ,
the stable limit cycles of

$$\dot{x} = y^2 - (x + 1) \left\{ (x - 1)^2 + a \right\} ,$$
$$\dot{y} = - x y ,$$

were determined. In [101] , these results were extended to the more general system
of the form

$$\dot{x} = P(y^2) - (x + 1) \left\{ (x - 1)^2 + a \right\} ,$$
$$\dot{y} = - x y.$$

In [102] , the existence of a unique, stable limit cycle for

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$$

is proved. In [103] , the sufficient conditions for the null solution of

$$\ddot{x} + (|\dot{x}| + q(x))\dot{x} + x - a^2 x^3 = 0$$

to be asymptotically stable were derived. The region of asymptotic stability was
also determined. In [104] , the existence of stable periodic solution of

$$\ddot{x}(t) + a b \dot{x}(t) + x(t - a) - c x^3(t - a) = 0$$

was proved for certain values of a, b, c.

In [105], the stability of oscillations in large chambers (hydraulics problems) was investigated.

In [106] , the uniform boundedness of solutions of

$$\ddot{x} + f(x, \dot{x}) \; \dot{x} + g(x) = h(t)$$

was considered.  In [107, 108, 109] , the boundedness of solutions of

$$\ddot{x} + \phi(\dot{x}) + \omega x = f(t)$$

was investigated.  In [110] , these results were extended to include the following system:

$$\ddot{x} + \phi(x, \dot{x}) + g(x) = f(t).$$

In [111] , the uniqueness and asymptotic stability of solutions of

$$\ddot{x} + f(t, x, \dot{x}) \; x + g(x) = h(t)$$

for particular f's and g's was proved.  In [112] , upper bound criteria and instability results were obtained for

$$\ddot{x} + f(x) \; \dot{x} + x = c \; \cos\left\{\frac{2\pi}{\omega} + a\right\}.$$

In [113] , stability conditions were derived for

$$\ddot{x} + x = \epsilon f(t, x, \dot{x}) ,$$

for small $\epsilon$.  In [114] , by approximate methods, the existence of stable persisting oscillations in an electrical circuit with an iron core and variable capacity was proved.  In [115] , the asymptotic behavior of the solutions of

$$t \ddot{x} + \dot{x} = \sin x$$

was investigated.  In [116, 117, 118] , conditions for the behavior of solutions in the neighborhood of a given bounded solution of

$$\dot{x} = f(t, x, y) ,$$

$$\dot{y} = g(t, x, y) ,$$

were derived.  In [119], the existence of a stable limit cycle for

$$\ddot{x} + f_1(x, \dot{x}) + g_1(x) = \epsilon F_1(x, \dot{x}, y, \dot{y})$$

$$\ddot{y} + f_2(y, \dot{y}) + g_2(y) = \epsilon F_2(x, \dot{x}, y, \dot{y})$$

was investigated. In [120], the stable oscillations of a nonlinear electrical network were studied. In [121], the uniform stability of solutions of

$$\dot{\underline{x}} = \underline{A}(t) \underline{x} + \underline{f}(t, \underline{x})$$

was studied in comparison with the linear system

$$\dot{\underline{x}} = \underline{A}(t) \underline{x}.$$

In [122], Conti studied the same systems and obtained results for boundedness, stability and continuation of solutions.

The next list of Italian references deals with the asymptotic behavior of solutions in the linear, time-varying case. (We list the reference number and the equation analyzed by the author.)

[123]      :      $\ddot{x} + q(t) x = 0.$

[124]      :      $a(t) \ddot{x} + b(t) \dot{x} + c(t) x = 0.$

[125]      :      $\dddot{x} + r(t) x = 0.$

[126, 127] :      $\dot{x} = A(t) x.$

[128, 129] :      $\dot{x} = A(t) x + a(t).$

(29) In [130], Putman considered the stability intervals of the Hill equations:

$$\ddot{x} + (a + f(t)) x = 0.$$

(30) In [131, 132], Liapunov theory was applied to the construction of limit cycles and the stability analysis of nonlinear feedback systems, respectively. In [133], Liapunov theory was applied to the study of modern automatic control theory as applied to the dynamics and control of nuclear rockets.

(31) In [134, 135], Levin and Nohel investigated the global asymptotic behavior of solutions in the case where the system has an integrable perturbation term. An example considered by the authors was:

$$\ddot{x} + h(t, x, \dot{x}) \dot{x} + f(x) = g(t),$$

where g(t) is the integrable perturbation term.  The analysis was based on the second method of Liapunov.  In [136] , <u>Yoshizawa</u> studied the relationships between the limiting sets of solutions of a perturbed equation and the solutions of an unperturbed equation through the use of Liapunov methods.  He considered the system

$$\dot{\underline{x}} = \underline{F} \ (t, \ \underline{x}) \ + \ \underline{G} \ (t, \ \underline{x}) \ ,$$

where the integrable perturbation term is $\underline{G}$ (t, $\underline{x}$); he also considered the system

$$\ddot{x} \ + \ f(x) \ \dot{x} \ + \ g(x) \ = \ h(t) \ ,$$

where his choice for the Liapunov function was

$$V \ (t, \ x, \ y) \ = \ \exp \left\{ -2 \int_0^t \ | \ h(t) \ | \quad dt \right\} \left\{ \int_0^x g(x) dx \ + \right.$$

$$\left. + \ \frac{1}{2} \ y^2 \ + \ 1 \right\} \ .$$

In [137] , <u>Markus</u> analyzed the system

$$\dot{\underline{x}} = \underline{H} \ (\underline{x}) \ + \ \underline{G} \ (t, \ \underline{x}) \ ,$$

where $\underline{G}$ is the perturbation term.  His analysis resembled that of the above authors. (32)  In [138] to [142] , the various authors analyzed the second order system defined by

$$\dot{x}_1 \ = \ f_1(x_1, \ x_2) \ , \quad f_1 \ (0, \ 0) \ = \ 0 \ ,$$

$$\dot{x}_2 \ = \ f_2(x_1, \ x_2) \ , \quad f_2 \ (0, \ 0) \ = \ 0 \ ,$$

By restricting the Jacobian matrix of $f_1$ and $f_2$ in various ways, the authors derived sufficient conditions for the asymptotic stability in the large of the null solution, investigated the orbital stability of bounded non-trivial solutions of the system, and considered certain boundedness properties of the solutions.  In [143] , the above results were extended to nonautonomous cases and to higher order systems.  Liapunov theory was used in some of proofs.

(33) In [144] , Liapunov theory was used to study the stability of a typical spin

generator described by the following equations:

$$\dot{x} = - ax + y \ ,$$

$$\dot{y} = - x - by (1 - k w) \ ,$$

$$\dot{w} = b \left\{ a (1 - w) - ky^2 \right\} \ ,$$

where a $>$ 0, b $>$ 0, c and k are constants.

(34) In [145] , the author discussed the stable sets in a perturbed system with

small perturbations, where he assumed that the set was asymptotically stable in the

unperturbed system. (He defines stability of a set in the paper.) Existence of

periodic solutions were also discussed. The methods used in the analysis were the

asymptotic fixed-point theorem of Browder and Liapunov functions.

(35) In [148] , the asymptotic relationship between two systems of ordinary differential

equations was discussed. If two systems of differential equations are <u>asymptotically</u>

<u>equivalent</u>, a solution of one system tends to a given solution of the other, and vise-

versa. One system can be called perturbed and the other one unperturbed. The author's

perturbed system was

$$\dot{x} = F (t, y, w) x + H (t, x, y, w) \ ,$$

$$\dot{y} = L (t, x, y, w) \ ,$$

$$\dot{w} = G (t, y) + M (t, x, y, w) \ ,$$

and the unperturbed system was

$$\dot{x} = F (t, y, w) x \ ,$$

$$\dot{y} = 0 \ ,$$

$$\dot{w} = G (t, y).$$

Liapunov functions are used in the analysis. This work is an extension of the work in

[147] and [146] . In [146], the systems considered were:

$$\underline{\dot{x}} = \underline{A} \ \underline{x} + \underline{p} (t) \ ,$$

$$\underline{\dot{x}} = \underline{A} \ \underline{x} + \underline{p} (t) + \underline{f} (t, \underline{x}) \ ,$$

where $\underline{f}$ is the perturbation term.

(36) In $\begin{bmatrix}149\end{bmatrix}$ , the author considered the boundedness of solutions of

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) ,$$

through the use of Liapunov functions. He then extended his theory to include

the following system:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) + \underline{g}(\underline{x}, t) ,$$

where $\underline{g}$ is a perturbation term.

## CONTRIBUTIONS TO THE FIELD OF STOCHASTIC STABILITY

<u>Vorovich</u>, $\begin{bmatrix}150\end{bmatrix}$ , was one of the earliest Russian authors who worked on the question

of stability in a stochastic system.

<u>Rosenbloom</u>, $\begin{bmatrix}151\end{bmatrix}$ , was the first to publish a paper in the U.S.A. on the subject

of stability of random systems. He considered first order differential equations

with random coefficients. For stationary Gaussian coefficients, for which the equation

may be integnated, <u>Rosenbloom</u> obtained a criterion for "stability in the mean" based

upon the value of the spectral function at zero frequency.

<u>Eringer and Samuels</u>, $\begin{bmatrix}152, 153, 154, 155\end{bmatrix}$ , considered a type of asymptotic stabili

that they referred to as mean square stability, which they applied in the studies of

certain higher order linear differential equations with random coefficients. Their

stability is defined only in terms of the second moments of the position components

of the state vector. They obtained explicit results for the case of Gaussian white-

noise coefficients. Their method of attack is based upon the integral equation associa

with the differential equation. In $\begin{bmatrix}156\end{bmatrix}$ , <u>Bogdanoff and Kozin</u> discuss and answer

some of the questions which we left unanswered in <u>Eringer's and Samuels'</u> work.

<u>Bertram and Sarachik</u>, $\begin{bmatrix}157\end{bmatrix}$ , defined stochastic versions of stability in the

phase space of random systems, relative to the three common modes of stochastic con-

vergence; that is, stability in probability, stability in the mean, and almost sure

stability. They proved sufficient conditions for the stability in the mean for general systems based upon the existence of Liapunov functions statisfying appropriate properties. They obtained Rosenblooms results and also obtained explicit results for higher order linear systems, where the coefficients were restricted to be bounded, finite-state, random coefficients.

Kats and Krasovskii, [158, 159] , considered the problem of stability via second method of Liapunov. They were interested in the construction of control functions that would stabilize systems subjected to random noise. Their general stability conditions and explicit results were of the same nature as those in [157] .

In references [160, 161, 162, 163], the authors were mainly interested in stochastic control theory. In [161] , Krasovskii and Lidskii searched for an optimum Liapunov function for a stochastic system. This Liapunov function guaranteed that the null solution of the system was "asymptotically stable in probability". In [163] , Lidskii is concerned with the study of control systems in which the transition process is described by means of stochastic linear differential equations. The construction of the Liapunov functions is accomplished by means similar to Chetaev's method. The system is subjected to the action of a random effect of the Markov type, developed during the control process, and also to disturbances which are random external impulsive disturbances. The paper considered the establishment of the control action of a control element which assumes statistical stability of a given motion with arbitrary initial deviations. Liapunov methods aided in this determination.

In [164, 165] , Khas'minskii considered the problem of the stability of a trajectory of a Markov process, with a different definition of stability. The necessary and sufficient condition for the stability of such processes is found, and it is analogous to the fundamental theorems of Liapunov's second method. The relationship between the stability of a system of ordinary differential equations and the stability of stochastic systems obtained by adding to the former a diffusion (random term) term is also investigated

In higher order cases, a sufficiently large diffusion term will reduce the stability —
this need not be true for order two or less.

In [166] , **Kozin** gave a simple, sufficient condition for almost sure stability (in the
Liapunov sense) of a class of linear systems with strictly stationary, metrically
transitive (i.e., satisfying the ergodic property) stochastic process coefficients.
The proof is based upon the fundamental Gronwall-Bellman lemma (see the "boundedness
section" of this report), of differential equation theory, and on the strong law of
large numbers for strictly stationary stochastic processes.

In references [167] to [172] , the work of **Caughey**, **Gray**, and others is reported.
In [167] , **Caughey** made note of the many errors which occurred in reference [153] .
In [168] , **Caughey and Dienes** showed that the behavior of linear dynamic systems, in
which a single parameter varies as a white-noise process, is an example of a continuous
multidimensional Markoff process. In [169] , **Caughey and Gray** were concerned with the
stability of the following system:

$$\dot{\underline{x}} = \left\{ \underline{A} + \underline{F}(t) \right\} \underline{x} ,$$

where $\underline{x}$ is an n-vector, **A** is a constant matrix, and $\underline{F}$ is a stochastic matrix.
The analysis was performed by a Liapunov approach. Sufficient conditions were derived
for almost sure stability of the null solution. The results were generalized to
include a certain class of non-linear system; and sufficient conditions were obtained
to guarantee the almost sure boundedness of the forced oscillations of linear dynamic
systems with stochastic coefficients. Examples were given for each of the above
problems.

In [171] , **Chelpanov** studied stability boundaries for second order random systems.
The correlation times of the random parameters were assumed to be much smaller than
the natural times of the system; thus, the random signals were assumed to be "white".
In [172] , **Gray** derived some sufficient stability conditions for linear systems with
random(non-white) coefficients. **Gray** used a generalized quadratic Liapunov function

in his analysis. He also gave several examples.

In [173], <u>Kats</u> studied the problem of "stability in probability" of stochastic systems in the large. A stability criterion based on the use of two Liapunov functions, due to Chetaev, was given. The Theorem proved for stochastic systems is analogous to that which was proved for ordinary differential equations by Barashin and Krasovskii. <u>Kats</u> also considered an example of his theorem.

In [174, 175, 176, 177], <u>Bershad, Tuel and Derusso</u> considered the stability of linear systems with randomly time-varying coefficients. However, they did <u>not</u> use Liapunov theory in their proofs.

In [178], <u>Vrkoc</u> gave an estimate of the probability that the solutions of a certain differential equation with random perturbations exceed a given bound.

In [179], <u>Kushner</u> considered some new theorems on the Liapunov theory of stochastic stability. The results were for the continuous time case, only. Many examples were given to illustrate the several techniques for determining and using stochastic Liapunov functions to obtain information about random trajectories. Also, useful bounds on the probability of certain important events were derived.

In [180], <u>Wonham</u> established sufficient conditions for recurrence and positivity for the diffusion process defined by a stochastic differential equation of Ito's type. He obtained conditions for non-recurrence and nonpositivity. The conditions required the existence of functions which closely resembled Liapunov functions. Thus, he was often able to infer "weak" stability of a stochastic system by starting with a Liapunov function for Lagrange stability of a corresponding deterministic system. Using this technique he discussed linear systems and a nonlinear system of Lure' type.

In [181], <u>Bucy</u> recognized the essential fact that the Liapunov functions in [158] were "nonnegative super martingales", and proved the first Liapunov theorem on almost sure convergence (with discrete time and global conditions). The theorems in [179] are based on sharper definitions and local conditions, and provide more useful information concerning random trajectories, than in [181].

## OTHER EXTENSIONS OF LIAPUNOV THEORY

**(A)** **Partial Differential Equations. (Infinite Degree of Freedom Systems)**

In the reference books, [182] and [183], _Zubov_ and _Hahn_ both discussed the stability of partial differential equations using a Liapunov function approach.

In [184], _Movchan_ considered the problem of stability of the "plane state" of a thin plate of infinite length, simply supported along two edges and subjected to the action of constant forces in its plane. The direct method of Liapunov was employed, thus causing the author to introduce an auxiliary metric space in order to construct in it the corresponding functionals needed in the analysis.

In [185 , 186], _Morchan_ considered other systems with an infinite number of degrees of freedom. In [185], the system was a dynamical system where deformations, temperatures and stresses were also considered in the stability problem. In [186], be considered a more general problem. For the processes considered he gave many definitions of stability; and for each one he proved a theorem of the direct method of Liapunov on the properties of the functionals which are necessary and sufficient for the existence of the particular type of stability or instability being considered. In some cases these functionals can also be used to prove uniqueness of solutions for certain partial differential equations.

In [187], _Kostandian_ considered the stability of the solution of the non-linear equation of heat conduction in the space $C_{L_2}$ . He employed the ideas contained in Liapunov's second method and arrived at sufficient conditions for asymptotic stability of the equilibrium solution. He mentioned that in references [188] and [189] other methods of stability analysis were presented, as well as extensive reference lists.

In [190], _Rakhmatullina_ investigated the stability of a somewhat more general problem than that studied in [187] . He used the methods which were first consider-ed in [191] to study the following equation:

$$\frac{\partial u}{\partial t} = \left\{ L u + f(t, \underline{x}, u) \right\},$$

where

$$L u \equiv \text{div} \left\{ \underline{A(\underline{x})}_T \ \underline{\text{grad } u} \right\} + \left\{ \underline{b(\underline{x})}_T \ \underline{\text{grad } u} \right\} +$$

$$+ c(\underline{x}) \ u.$$

is an elliptic operator. One method used was that of "differential inequalities" (see the boundedness section of the report), which are related to the Liapunov methods.

In [192], <u>Slobodkin</u> considered the stability of some simple systems from the linear theory of elasticity. He used the direct method of Liapunov to establish his results. The Liapunov funtional used in the analysis was related to the total energy of the system being considered.

In [193, 194, 195, 196], <u>Lakshmikantham</u> considered the problem of stability of solutions of parabolic equations and certain functional equations by employing the direct method of Liapunov. He obtained a number of results, in a unified way, by using these techniques. For instance, he found the stability of the solutions, and investigated certain examples. He also indicated that <u>Liapunov - like vector functions</u> were useful in some cases.

B) <u>Differential - Difference Equations and Functional Equations</u>

In reference [183], <u>Hahn</u> considered some of the results from the stability studies of systems with a time-lag.

In reference [197], <u>Krasovskii</u> considered the general definitions and theorems of Liapunov's second method for equations with time-lags. The systems considered in his book were linear systems with time-lag, nonlinear systems, integrodifferential equations with time lags, and systems with persistent disturbances. He also gave methods to construct the Liapunov functionals for special systems.

In reference [198], <u>Bellman and Cooke</u> considered the field of differential-difference equations in great detail. Some of the topics covered which are

pertinent to this report are: small perturbation theory, definitions of various types of stability, existence theorems, uniqueness theorems, asymptotic behavior of solutions, stability theorems, time-varying, time-lag systems, Liapunov functions and functionals, and a reference list, which will not be repeated here. In reference [199], Bellman and Danskin gave an earlier survey of the field of differential-difference equations, the survey being interested mainly in stability.

In [200, 201], Razumikhin considered the stability of systems with time-lag by employing the direct method of Liapunov. In [201], he obtained his results by considering the first approximations for the systems. Also, he applied his theory to the following practical example, from [202]:

$$\ddot{x}(t) + a_1\dot{x}(t) + a_2x(t) + a_3x(t-\tau) = 0,$$

which describes the transient processes in a certain automatic control system.

In [203], Shimanov proved that the known theorems of Liapunov and Chetaev concerning stability can be extended to systems with retardation. Also, he gave a criterion of instability in the first approximation of the motion of systems with retardation. The equations of perturbed motion which he studied are given by

$$\dot{x}(t) = f(x(t+\tau), t),$$

where $f(0, t) \equiv 0$. These equations are called "equations with after-effects".

In [204], Shimanov gave a practical method for solving the problem of the undisturbed motion of a system with time-lag for the critical case when one of the roots is zero. Liapunov theory was used in the analysis. The author considered two particular second-order nonlinear, time-lag systems as examples for his procedure.

In [205], Popov and Halanay considered an application of Popov's method (see "Control Section" of this report) to the problem of stability of some systems

with lagging arguments. The form of the equations studied are as follows:

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{x}\ (t - \tau) + \underline{L}\ \phi\{\sigma\ (t-\tau)\}$$

$$\sigma = \underline{c}_T\ \underline{x}\ ,$$

where $\underline{A}$, $\underline{B}$, $\underline{L}$, $\underline{c}$ are constants and $\phi$ and $\sigma$ are scalars. A special case of this equation was studied in [206].

In [209], Wang and Bandy have shown that time-delayed variables can enter into distributed - parameter processes due to the presence of both internal and external delayed action energy sources. The dynamic behavior of such processes were described by a system of partial differential-difference equations. The authors paid particular attention to the class of equations which admitted product solutions so that their time-dependent equations were reducible to a denumerable infinity of ordinary differential-difference equations. Motivated by the theory developed in [207] and [208], the authors gave an extended version of Liapunov's stability theory for such systems. Its application was illustrated by the study of a one-dimensional diffusion process with non-linear delayed-action sources. In [210], Wang considered the asymptotic stability of the equilibrium states of a nonlinear diffusion system with time delays. He also gave a physical interpretation of his problem; namely, an automatically controlled furnace with time-delays, used for heat-treatment of certain materials.

In [211], Hale considered the existence of periodic and almost periodic solutions of linear nonhomogeneous functional-differential equations under a hypothesis of uniform asymptotic stability for homogeneous systems, (as defined in his paper). He unified and generalized the results which others had obtained for differential-difference equations by a systematic use of a special Liapunov functional. He considered, as applications, two specific types of nonlinear systems. Also, Hale gave an extensive bibliography: concerning the above problems.

In [212], <u>Hale</u> discussed linear functional-differential equations with constant coefficients. The topics discussed were: eigenspaces, use of the adjoint equation, perturbation theorems, and the geometric theory of differential-difference equations.

In [213], <u>Hale</u> pointed out that Krasovskii proved, in [197], that the stability theorems of Liapunov and their converses can be extended to differential equations with delayed arguments if the equations are discussed in a space of continuous functions C, over a finite interval. These theorems hold if the Liapunov functions are also defined over this space C. These yield necessary and sufficient conditions for stability and thus are not too useful for applications. For this reason, <u>Hale</u> determined conditions, by means of Liapunov functionals, which were only sufficient for stability. This took the form of generalizations of the work of LaSalle, [214], for ordinary differential equations. <u>Hale</u> considered as an example a problem due to Levin and Nohel, discussed earlier in this section.

In [215], <u>Hale</u> continued the work which he discussed in [207] and [216] to a more general type of equation, namely, a functional-differential equation. Liapunov functionals were used throughout the discussion.

In [217], <u>Hale</u> continued his discussion on the use of Liapunov functionals to investigate asymptotic stability of a certain class of functional-differential equations.

In [218], <u>Seifert</u> gave stability conditions on the bounded solutions of systems with "almost periodic" time dependence which will guarantee separation, defined in the paper, of one such solution with respect to the others. Then, he obtained, in terms of such stability conditions, an existence theorem for almost periodic solutions of almost periodic systems.

In [219] , <u>Seifert</u> used Liapunov functions to derive sufficient conditions

for uniform stability of almost-periodic solutions of almost-periodic systems

of differential equations.

In [220] and [221] , <u>Stokes</u> and <u>Jones</u> used fixed-point theorems to study the

problems of stability of nonlinear systems and functional-differential systems,

respectively.

In [222] , <u>Taam</u> obtained conditions for certain nonlinear differential

equations in a Banach space to possess a unique almost periodic solution which

is positively asymptotically stable and negatively unstable.

In [223] , <u>Jones</u> studied the existence of periodic motions in Banach Spaces

and their applications to functional-differential equations.  He employed the

asymptotic fixed point theorems in his work; and he also considered many

interesting physical systems as examples of his theory.

In [224] , <u>Stokes</u> proved that if the associated linear variational functional-

differential equation has only one non-trivial one-parameter family of periodic solu-

tions, and all the remaining solutions tend to zero, than the limit cycle of the

autonomous functional-differential equation is orbitally asymptotically stable;

with asymptotic phase.

In [225] , <u>Hale and Perello</u> took another step in the geometric direction to

extend results known for ordinary differential equations to nonlinear functional-

differential equations of the finite lag type.  They used Liapunov functionals

in some of their work.

In [226] , <u>Hale</u> proved sufficient conditions for stability and instability

of autonomous functional-differential equations.  Liapunov functionals were used.

Many practical examples were considered:

stability of a circulating fuel nuclear reactor, Volterra's integrodifferential

equation, viscoelasticity problems, and many others.


C.  Topological Dynamics and Generalized

Dynamic Systems

Under this "item" we will just list some of the references in the field

which either define what a generalized dynamic system really is, or which make

use of Liapunov functions in the study of stability in generalized dynamic

systems.  The references are numbered from [227] to [239].  Also, in

references [182] and [183], we can find discussions on the stability of

invariant sets of certain dynamic systems.

## REFERENCES

1)  Pliss, V. A., "An Investigation of a Nonlinear System of three Differential Equations," <u>Dokl. Ak. Nauk USSR</u>, Vol. 117, pp 184-187, 1957.

2)  Ogurtsov, A. I., "The Stability of Solutions of two Nonlinear Differential Equations of the Third and Fourth Orders," <u>P.M.M.</u>, Vol. 23, No. 1, pp 179-181, 1959.

3)  Ezeilo, J.O.C., "On the Boundedness and the Stability of Solutions of Some Differential Equations of Fourth Order", <u>Journal of Math. Anal. and Appl.</u>, Vol. 5, p 136-146, 1962.

4)  Ezeilo, J.O.C., "Some Results for the Solutions of a Certain System of Differential Equations", <u>Jour. of Math. Anal. and Appl.</u>, Vol. 6, pp 387-393, 1963.

5)  Ezeilo, J.O.C., "A Stability Result for Solutions of a Certain Fourth Order Differential Equation", <u>Jour. London Math. Soc.</u>, Vol. 37, pp 28-32, 1962.

6)  Ezeilo, J.O.C., "On the Stability of Solutions of Certain Differential Equations of the Third Order", <u>Quant. Jour. of Math</u>, Oxford, Ser. 2, Vol. II, pp 64-69, 1960.

7)  Bellman, R., "Vector Liapunov Functions", <u>SIAM Jour. on Control</u>, Ser. A. Vol. I., No. 1, pp 32-34, 1962.

8)  Leighton, W., "On the Construction of Liapunov Functions for Certain Autonomous Nonlinear Differential Equations", <u>Contr. to Diff. Eq.</u>, Vol. II, No. 1 - 4, pp 367-383, 1963.

(9)  LaSalle, J. P. and Lefschetz, S., <u>Stability by Liapunov's Direct Method, With Applications</u>, Academic Press, New York, 1961.

(10) Skidmore, A. S., "On the Stability of Solutions of a Differential Equation of Fourth Order", <u>Report Submitted to U. S. Army Res. Off.</u>, Western Reserve University, Cleveland, Ohio, 1965.

(11) Krasovskii, N. N., <u>Stability of Motion</u>, Stanford University Press, Stanford, Cal., p 69, 1963.

(12) Leighton, W., "Morse Theory and Liapunov Functions", to appear in <u>Rend. del Circolo Matematico di Palermo</u>.

(13)  Cartwright, M. L., "On the Stability of Solutions of Certain Differential Equations of the Fourth Order," Quaint. J. Mech. and Appl. Math, Vol. 9, pp 185-194, 1956.

(14)  Sansone, G., "Nonlinear Differential Systems of the Third and Fourth Order", Diff. Eq. and Their Appl., Proc. of Prague Conf., Academic Press, pp 143-165, 1962.

(15)  Simanov, S. N., "On the Stability of Solutions of a Nonlinear Equation of Third Order", PMM, Vol. 17 (Russian(, pp 369-372, 1953.

(16)  Krasovskii, N. N., "On the Stability of the Solutions of a Nonlinear System of Three Equations for Arbitrary Initial Disturbances", PMM, Vol. 17 (Russian), pp 339-350, 1953.

(17)  Tusov, A. P., "Stability Problems for a Certain Regulating System", Vestnik Leningrad. Univ., No. 2, (Russian), pp 43-70, 1955.

(18)  Tusov, A. P., "On the Stability in the Large of a Certain Regulation System", Vestnik Leningrad Univ., No. 12, (Russian), pp 57-75 - 1957.

(19)  Tusov, A. P., "Necessary and Sufficient Conditions for the Stability in the Large of a System of Regulation," Dokl. Akad. Nauk. SSSR 4 (Russian(, pp 101-105, 1960.

(20)  Pliss, V. A., "Investigation of a Nonlinear Differential Equation of the Third Order", Dokl. Akad. Nauk SSSR (N.S.), (Russian), Vol. 111, pp 1178-1180, 1956.

(21)  Pliss, V. A., "Necessary and Sufficient Conditions for the Stability in the Large

of a Homogeneous System of Three Differential Equations", <u>Dokl. Akad. Nauk SSSR (N.S.)</u> (Russian(, Vol. 120, pp 708-710, 1958.

(22)   Pliss, V. A., "Aizerman's Problem in the case of Three Simultaneous Differential Equations", <u>Dokl. Akad. Nauk. SSSR (N.S.)</u> (Russian)Vol. 121, pp 422- 425, 1958.

(23)   Vaisbord, E.M., "On the Existence of a Periodic Solution and the Boundedness in the Large of the Solution of a Differential System of the Third Order", <u>Izv. Vyss. Uc. Zaved, Matematika</u> (Russian), Vol. 4 (11) pp 38-49, 1959.

(24)   Ogurcov, A. I., "On the Stability of Certain Nonlinear Differential Equations of the Third and Fourth Orders," <u>Izo. Vyss. Uc. Zaved. Matematika</u>, (Russian) Vol. 3, 10, pp 200-209, 1959.

(25)   Tabueva, V.A., "Conditions for the Existence of a Periodic Solution of a Third-Order Differential Equations", <u>PMM.</u> Vol. 25, (Russian), pp. 961-962, 1961.

(26)   Ezeilo, J.O.C., "On the Boundedness of Solutions of Certain Differential Equations of Third Order," <u>Proc. London Math. Soc.</u>, No. 3, Vol. 9, pp 74-114, 1959.

(27)   Ezeilo, J.O.C., "An Elementary Proof of a Boundedness Theorem for a Certain Third-Order Differential Equation" <u>J. London Math. Soc.</u>, Vol. 38, pp 11-16, 1963.

(28)   Ezeilo, J.O.C., "On the Existence of Periodic Solutions of a Certain Third-Order Differential Equation," <u>Proc. Cambridge Philos. Soc.</u>, Vol. 56, pp 381-389, 1960.

(29)   Ezeilo, J.O.C., "A Property of the Phase-Space Trajectories of a Third-Order Nonlinear Differential Equation," <u>J. London Math. Soc.</u>, Vol. 37, pp 31-41, 1962.

(30)   Ezeilo, J.O.C., "A Boundedness Theorem for a Certain Third-Order Differential Equation", <u>Proc. London Math. Soc.</u>, No. 3, Vol. 13, pp 99-124, 1963.

(31)   Ezeilo, J.O.C., "A Boundedness Theorem for Some Nonlinear Differential Equations of the Third Order," <u>J. London Math. Soc.</u>, Vol. 37, pp 469-474, 1962.

(32)   Pliss, V.A., "Boundedness of Solutions of Certain Nonlinear Third-Order Differential Equations," <u>Dokl. Akad. Nauk. SSSR 139</u>, (Russian), pp 304, 1961.

(33) Skackov, B.N., "Qualitative Picture of the Behavior of Integral Curves in the Neighborhood of a Singular Point in One Case", <u>Vestrik Leningrad Univ. 9</u>, (Russian), pp 65-69, 1954.

(34) Skackov, B.N., "Questions of Stability in the Large and Relation Qualities for Certain Systems of Differential Equations," <u>Vestrik Leningrad Univ. 12, 19</u>, (Russian), pp 35-46, 1957.

(35) Skackov, B.N., "On the Stability of a Class of Nonlinear Systems of Automatic Control," <u>Vestrik Leningrad Univ. 12, 1</u>, (Russian), pp 46-56, 1957.

(36) Skackov, B. N., "Questions of Stability in the Large and Regulating Properties for a Certain System of Differential Equations," <u>Vestrik Leningrad Univ. 13, 3</u> (Russian), pp 67-80, 1957.

(37) Duffin, R.J., "Nonlinear Networks. I", <u>Bull. of Amer. Math. Soc.</u>, Vol. 52, No. 10, pp 833-838, 1946.

(38) Duffin, R. J., "Nonlinear Networks. IIa," <u>Bull. of Amer. Math. Soc.</u>, Vol. 53, No. 10, pp 963-971, 1947.

(39) Duffin, R.J., "Nonlinear Networks. IIb," <u>Bull. of Amer. Math. Soc.</u>, Vol. 54, No. 2, pp 119-127, 1948.

(40) Duffin, R.J., "Nonlinear Networks. III", <u>Bull. of Amer. Math. Soc.</u>, Vol. 55, No. 2, pp 119-129, 1949.

(41) Duffin, R.J., "Nonlinear Networks. IV", <u>Proc. of Amer. Math. Soc.</u>, Vol. 1, No. 2, pp 233-240, 1950.

(42) Jones, J. Jr., "On Nonlinear Second Order Differential Equations", <u>Proc. of Amer. Math. Soc.</u>, Vol. 9, No. 4, pp 586-588, 1958.

(43) U + , W.R., "Properties of Solutions of Second Order Nonlinear Differential Equations", <u>Proc. of Amer. Math. Soc.</u>, Vol. 8, No. 6, pp 1024-1028, 1957.

(44) Levin, J. J., "The Asymptotic Behavior of the Solution of a Volterra Equation", Proc. of Amer. Math. Soc., Vol. 14, No. 4, pp 534-541, 1963.

(45) Levin, J. J. and Nohel, J. A., "Note on a Nonlinear Volterra Equation," Proc. of Amer. Math. Soc., Vol. 14, No. 6, pp 924-929, 1963.

(46) Kemp, R. R. D., "Remarks on Systems of Nonlinear Volterra Equations", Proc. of Amer. Math. Soc., Vol. 14, No. 6, pp 961-962, 1963.

(47) Nohel, J. A., "Some Problems in Nonlinear Volterra Integral Equations," Bull. Amer. Math. Soc., Vol. 68, pp 323-330, 1962.

(48) Widder, D. V., The Laplace Transform, Princeton Univ. Press, Princeton, N. J.

(49) Bhatia, N. P., "On Exponential Stability of Linear Differential Systems", Jour. of the Soc. for Industrial & Appl. Math., Ser. A Control, Vol. 2, No. 2, pp 181-191, 1964.

(50) Malkin, I. G., "On the Construction of Liapunov Functions for Systems of Linear Equations," PMM, Vol. 16, pp 239-242, 1952.

(51) Roseau, M., "Sur la second methode de Liapounoff", C. R. Acad. Sci. Paris, Vol. 252, pp 2056-2057, 1961.

(52) Roseau, M., "Sur la stabilite de la solution x = 0 du systeme differential $\dot{x} = A(t) x + f(t,x)$," J. de Math, Paris, Vol. 41, No. 3, pp 201-212, 1962.

(53) Rosenbrock, H. H., "A Method of Investigating Stability", Proc. I.F.A.C. Congress, Basle, 1963.

(54) Rosenbrock, H. H., "Some Conditions for the Stability of Nonlinear time - Dependent Differential Equations," J. of S.I.A.M., Ser. A. Control, Vol. 2, No. 2, pp 171-180, 1964.

(55) Rosenbrock, H. H., "On the Stability of a Second-Order Differential Equation," London Math. Soc., Vol. 39, pp 77-80, 1964.

(56) Brayton, R. K. and Moser, J. K., A Theory of Nonlinear Networks, I B M Res. Paper RC - 906, March 20, 1963, Yorktown Heights, N. Y.

(57) Hochstadt, H., "On the Stability of Certain Second Order Differential Equations", J. of SIAM Vol. 12, No. 1, pp 58-59, 1964.

(58)  Bebernes, J. W. and Vinh, N. X., "On the Asymptotic Behavior of Linear Differential Equations," Amer. Math. Monthly, Vol. 72, No. 3, pp 285-287, 1965.

(59)  Struble, R. A., "A Note on Damped Oscillations in Nonlinear Systems", SIAM Review, Vol. 6, No. 3, pp 257-259, 1964.

(60)  LaSalle, J. P. and Wonham, W. M., "The Second International Congress of IFAC on Automatic Control," SIAM - Review, Vol. 6, No. 3, pp 296-309, 1964.

(61)  Wong, J.S.W., "A Note on Boundedness Theorems of Certain Second Order Differential Equations", SIAM - Review, Vol. 6, No. 2, pp 175-176, 1964.

(62)  Zubov, V. I., "On the Behavior of Integral Curves in the Neighborhood of a Periodic Motion", PMM, Vol 25, No. 2, pp 303-313, 1961.

(63)  Matrosov, V. M., "On the Stability of Motion" PMM, Vol. 26, No. 5, pp 885-895, 1962.

(64)  Matrosov, V. M., "On the Theory of Stability of Motion," PMM, Vol. 26, No. 6, pp 992-1002, 1962.

(65)  Chzhan Sy-in, "On the Estimates of the Solutions of Systems of Differential Equations of the Accumulation of Disturbances and the Stability of Motion Over a Finite Time Interval," PMM, Vol. 23, No. 4, pp 640-649, 1959.

(66)  Lebedev, A. A., "On the Stability of Motion on a Given Time Interval," PMM, Vol. 18, No. 1, 1954.

(67)  Lebedev, A. A., "On the Problem of the Stability of Motion over a Finite Time Interval", PMM, Vol. 18, No. 1, 1954.

(68)  Karacharov, K. A. and Piliutik, A. G., "Stability of Nonstabilized Motion on a Finite Time Interval, Part I." Otdel-nauchno-tekhnicheskoi informatsii, 1958.

(69)  Chzhan Sy-In, "On the Stability of Motion Over a Finite Time Interval", PMM, Vol. 23, No. 2, 1959.

(70)  Zubov, V. I., "On a Method of Investigating the Stability of a Null-Solution in Doubtful Cases", PMM, Vol. 22, No. 1, pp 46-49, 1958.

(71)  Razumikhin, B. S., "On the Application of the Liapunov Method to Stability Problems," PMM, Vol. 22, No. 3, pp 338-349, 1958.

(72)  Aizerman, M. A. and Gantmaker, F. R., "On the Stability of Periodic Motions" PMM, Vol. 22, No. 6, pp 750-758, 1958.

(73)  Angronov, A. A. and Vitt, A. A., "On the Stability According to Liapunov", Coll. Works of Andronov, Izd. Akad. Nauk SSSR, pp 140-141, 1956.

(74)  Livartovskii, I. V., "The Stability of a Solution of a System of Differential Equations with Discontinuous Right-Hand Sides", PMM, Vol. 23, No. 3, pp 598-603, 1959.

(75)  Kalinin, S. V., "The Stability of Periodic Motions in the Case of two Zero Roots", PMM, Vol. 23, No. 5, pp 975-977, 1959.

(76)  Chetaev, N. G., "The Problem of Klein", PMM, Vol. 24, No. 1, pp 23-32, 1960.

(77)  Aizerman, M. A. and Gantmakher, F. R., "On the Stability of the Equilibrium Positions for Discontinuous Systems", PMM, Vol. 24, No. 2, pp 283-293, 1960.

(78)  Solntsev, Iu. I., "On the Liapunov Stability of the Position of Equilibrium of Systems of Two Differential Equations with Discontinuous Right-Hand Sides," MGU Uch. Zap Matematika, Vol. 4, No. 148, 1951.

(79)  Boltianskii, V. G. and Pontriagin, L. S., "On the Stability of the Equilibrium Positions of Relay Systems of Ordinary Differential Equations," Tr. 3-Vo Vsesoyzn . Matema s"ezda, t . 1 Izd - Vo Akad. Nauk SSSR, 1956.

(80)  Anosov, D. V., "On the Stability of the Positions of Equilibrium of a Relay System," Aotomatika i Telemehanika, No. 2, 1959.

(81)  Neimark, Iu. I. and Kiniapin, S. D., "On the Stability of the Positions of Equilibrium of Relay Systems", Aotomatika i Telemekhanika, No. 9, 1959.

(82)  Kuzimin, P.A., "Quadratic Integrals of Linear Mechanical Systems", PMM, Vol. 24, No. 3, pp 575-577, 1960.

(83)  Markhashov, L. M., "On the Critical Cases of Stability of Stationary Motions According to Liapunov," PMM, Vol. 25, No. 2, pp 265-275, 1961.

(84) Klimushchev, A. I. and Krasovskii, N. N., "Uniform Asymptotic Stability of Systems of Differential Equations with a Small Parameter in the Derivative Terms," PMM, Vol. 25, No. 4, pp 680-690, 1961.

(85) Regish, M., "On Nonuniform Asymptotic Stability", PMM, Vol. 27, No. 2, pp 231-243, 1963.

(86) Krein, M. G., "On Certain Problems on the Maximum and Minimum of Characteristic Values and on the Lyapunov Zones of Stability," trans. from the Russian by the Amer. Math Soc., Ser. 2, Vol. 1, pp 163-187.

(87) Hale, J. K. and Stokes, A. P., "Conditions for the Stability of Nonautonomous Differential Equations", Jour. of Math. Anal. and Appl., Vol. 3, pp 50-69, 1961.

(88) Fowler, T. K., "Lyapunov's Stability Criteria for Plasmas," Jour. of Math. Physics Vol. 4, No. 4, Apr. 1963.

(89) Donalson, D. D. and Leondes, C. T., "A Model Referenced Parameter Tracking Techniqu for Adoptive Control Systems, II. Stability Analysis by the Second Method of Lyapunov," IEEE Trans., pp 252-262, Sept. 1963.

(90) Donaldson, D. D., "The Theory and Stability Analysis of a Model Referenced Paramete Tracking Technique for Adaptive Automatic Control Systems", Ph. D. Dissentation - U. of C Los Angeles, Calif., June 1961.

(91) Tomovie, R., Sensitivity Analysis of Dynamic Systems, Trans. by D. Tornquist, McGraw-Hill Book Co. Inc., 1963.

(92) Jaffe, R. C., "Liapunov Stability and Optimal Control," Grummon Aircraft Eng. Corp R. N. 154., 1964.

(93) Vogt, W. G., "Relative Stability via the Direct Method of Liapunov," ASME Ser. D. Mar. 1964.

(94) Seifert, G., "Rotated Vector Fields and an Equation for Relaxed Oscillations", Contr. to the Theory of Nonlinear Oscillations, Vol. IV, pp 125-139, 1959.

Note

The next several references, [95] to [129] , form a partial list of the contributions to stability theory due to the Italian Mathematicians. The journals are listed but the titles of the papers are not given. In the text of this section, the outlines of the papers are presented.

(95)  Sestini, G., Quaderno Scient. Convegno Mat. Modena, pp 3-18, 1952.

(96)  Ghizzetti, A., Annali di Mat. pura ed appl., Vol. 51, No. 4, pp 167-202, 1960.

(97)  Manaresi, G., Boll. Un. Mat. Ital, Vol. 10, No. 3, pp 537-540, 1955.

(98)  Manaresi, G., Rend. Accad. Sci. Bologna Vol. 2, No. 11, pp 184-189, 1954-1955.

(99)  Gagliardo, E., Boll. Un Mat. Ital Vol. 8, No. 3, pp 309-314, 1953.

(100)  Sansone, G. and Conti, R., Annali di Mat. pura. ed. appl., Vol. 37, No. 4, pp 37-59, 1954.

(101)  Virgopia, N., Rend. di. Mat. e sue. appl. Vol. 15, No. 5, pp 153-176, 1956.

(102)  De Castro, A., Boll. Un. Mat. Ital., Vol. 9, No. 3 pp 369-372, 1954.

(103)  Neppi Modona, L., Boll. Un. Mat. Ital., Vol. 8, No. 3, pp 428-441, 1953.

(104)  Colombo, G., Annali Univ. Ferra, (N.S.) Sez. VII, Sci. Mat. 4, pp 35-50, 1955.

(105)  Evangelisti, E., L'energia elettrica Vol. 28, 1951.

(106)  Manfredi, B., Boll. Un Mat. Ital, Vol. 11, No. 3, pp 64-71, 1956.

(107)  Sestini, G., Atti, IV Congr. Un. Mat. Ital. Taormina, pp 559-564, 1951.

(108)  Amerio, L., Annali. di. Mat. pura. ed. appl., Vol. 39, No. 4., pp 97-119, 1955.

(109)  Sestini, G., Rivista Mat. Univ. Parma Vol. 2., pp 303-314, 1951.

(110)  Sestinei, G., Rivista Mat. Univ. Parma Vol. 5, pp 227-232, 1954.

(111)  De Castro, A., Rivista, Mat. Univ. Parma No. 4., pp 133-143, 1953.

(112)  Graffi, D., Rivista. Mat. Univ. Parma Vol. 3, pp 317-326, 1952.

(113)  Graffi, D., Annali di Mat. pura ed appl., Vol. 49, No. 4, pp 326-358, 1960.

(114)  Manacorda, T., Boll. Un. Mat. Ital. Vol. 8, No. 3, pp 281-285, 1953.

(115)  Sansone, G., <u>Annali di Mat. pura. ed. appl.</u> Vol. 50, No. 4, pp 439-466, 1960.

(116)  Amerio, L., <u>Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fi's. Mat. Nat.</u>, Vol. 10,

No. 8, pp 206-212, & pp 289-297, 1951.

(117)  Amerio, L., <u>Rivista, Matem. Univ. Parma.</u>, Vol. 3, pp 207-231, 1952.

(118)  Amerio, L., <u>Commun. on Pure and Appl. Math.</u> Vol. 6, pp 435-454, 1953.

(119)  Columbo, G., <u>Rend. Sem. Mat. Univ. Padova</u>, Vol. 24, pp 400-420, 1955.

(120)  Vignoli, P., <u>Rend. Sem. Mat. Univ. Padova</u>, Vol. 27, pp 375-386, 1957.

(121)  Santoro, P., <u>Atti Accad. Naz. Lincei. Rend Cl. Sci. fis. mat. nat.</u>, Vol. 28,

No. 8, pp 336-341, 1960.

(122)  Conti, R., <u>Boll. Un. Mat. Ital.</u>, Vol. 11, No. 3, pp 344-349, 1956.

(123)  Tricomi, F. G., <u>Atti. Accad. Naz. Lincei. Rend. Cl Sci. fis. mat. nat.</u>,

Vol. 17, No. 8 pp 137-141, 1954.

(124)  Salinas, B. R., <u>Rend. Circolo. Mat. Palermo</u>, Vol. 8, No. 2, pp 206-224, 1959.

(125)  Villari, G., <u>Boll. Un. Mat. Ital.</u>, Vol. 13, No. 3, pp 73-78, 1958.

(126)  Conti, R., <u>Atti. Accad. Naz. Lincei, Rend. Cl. Sci. fi's. mat. nat.</u>, Vol. 19,

No. 8, pp 247-250, 1955.

(127)  Barbuti, U., <u>Annali Scuola Nom. Sup. Pisa</u> Vol. 10, No. 3, pp 185-215, 1956.

(128)  Manacorda, T., <u>Boll. Un. Mat. Ital.</u> Vol. 7, No. 3, pp 281-284, 1952.

(129)  Conti, R., <u>Rivista. Mat. Univ. Parma</u>, Vol. 6, pp 3-35, 1955.

(130)  Putnam, C. R., "On the Stability Intervals of the Hill Equation," <u>J. of SIAM</u>,

Vol. 7, No. 1, pp 101-106, 1959.

(131)  Goldwyn, R. M. and Cox, K. J., "Limit Cycle Construction Using Liapunov

Functions," <u>IEEE Trans. Auto. Control</u>, Vol. AC 10, No. 1, pp 97-99, 1965.

(132)  Willems, J.L.M.A. and Brockett, R. W., "Remarks on the Stability of Nonlinear

Feedback Systems", <u>IEEE Trans. Auto. Control</u>, Vol. AC 10, No. 1, p 104, 1965.

(133)  Melsa, J. L. and Schultz, D. G., <u>Research in and Application of Modern Automatic

Control Theory to Nuclear Rocket Dynamics and Control</u>, Semi-Annual Status Report, Vol. II

University of Arizona, March 1965.

(134)  Levin, J. J. and Nohel, J. A., "Global Asymptotic Stability for Nonlinear Systems of Differential Equations and Applications to Reactor Dynamics", <u>Arch. Rational Mech. Anal.</u>, p 194, Vol. 5, 1960.

(135)  Levin, J. J., "On the Global Asymptotic Behavior of Nonlinear Systems of Differential Equations," <u>Arch. Rotional Mech. Anal.</u>, Vol. 6, p 65, 1960.

(136)  Yoshizawa, T., "Asymptotic Behavior of Solutions of a System of Differential Equations", <u>Contr. to Diff. Eq.</u> Vol. 1, No. 3, pp 371-387, 1963.

(137)  Markus, L., "Asymptotically Autonomous Differential Systems", <u>Contr. to the Theory of Nonlinear Oscillations</u>, Vol. III, Annals of Math. Studies, No. 36, Princeton University Press, Princeton, New Jersey, pp 17-19, 1956.

(138)  Markus, L. and Yamabe, H., "Global Stability Criteria for Differential Systems," <u>Osaka Math. J.</u> Vol. 12, pp 305-317, 1960.

(139)  Hartman, P., "On the Stability in the Large for Systems of Ordinary Differential Equations", <u>Can. J. Math.</u>,Vol. 13, pp 408-492, 1961.

(140)  Krasovski, N. N., "Some Problems of the Stability Theory of Motion" (Russian), <u>Gosudartv, Izdat. Fiz. Mat. Lit., Moscow</u>, 1959.

(141)  Olech, C., "On the Global Stability of an Autonomous System on the Plane," <u>Contr. to Diff. Eq.</u>, Vol. 1, No. 3, pp 389-400, 1963.

(142)  Borg, G., "A Condition for the Existence of Orbitally Stable Solutions of Dynamic Systems," <u>Kungl. Tekn. Hogsk. Handl.</u>, Vol. 153, 12 pp, 1960.

(143)  Hartman, P. and Olech, C., "On Global Asymptotic Stability of Solutions of Differential Equations," <u>Trans. of Amer. Math. Soc.</u>, Vol. 104, No. 1, pp 154-178, 1962.

(144)  Sherman, S., "A Third-Order Nonlinear System Arising from a Nuclear Spin Generator", <u>Contr. to Diff. Eq.</u> Vol. II, pp 197-227, 1963.

(145)  Yoshizawa, T., "Stable Sets and Periodic Solutions in a Perturbed System", <u>Contr. to Diff. Eq.</u>, Vol. II, pp 407-420, 1963.

(146) Brauer, F., "Asymptotic Equivalence and Asymptotic Behavior of Linear Systems, Michigan Math. J., Vol. 9, pp 33-43, 1962.

(147) Kato, J., "The Asymptotic Behavior of the Solutions of Differential Equations on the Product Space," Jap. J. Math., Vol. 32, pp 51-85, 1962.

(148) Kato, J., "The Asymptotic Relations of Two Systems of Ordinary Differential Equations", Contr. to Diff. Eq.,Vol. III, No. 2, pp 141-161, 1964.

(149) Ezeilo, J. O. C., "On the Existence of an Almost Periodic Solution of a Non-Linear System of Differential Equations", Contr. to Diff. Eq., Vol. III, No. 3, pp 337-349, 1964.

(150) Vorovich, I. I., "On the Stability of Motion Under Random Disturbances", IZV. Akad. Nauk. SSSR, ser. matem. Vol. 20, No. 1, 1956.

(151) Rosenbloom, A., "Analysis of Linear Systems with Randomly Time - Varying Parameters Proc; of the Symp. on Inf. Networks, New York 1954, Brooklyn Polytechnic Press, Brooklyn(1 pp 145-153.

(152) Samuels, J. C. and Eringen, A. C., "On Stochastic Linear Systems", Jour. of Math. a Phys., Vol. 38, pp 83-103, 1959.

(153) Samuels, J. C., "On the Stability of Random Systems and the Stabilization of Deterministic Systems with Random Noise", J. Acoust. Soc. Amer., Vol. 32, pp 594-601, 196(

(154) Samuels, J. C. "Theory of Stochastic Systems with Gaussion Parameter Variations", J. Acoust. Soc. Amer., Vol. 33, pp 1782-1786, 1961.

(155) Samuels, J. C., "On the Mean-Square Stability of Random Linear Systems", Proc. Int. Syms. on Circuit and Inf. Theory, Los Angeles, Calif. IRE Trans. on Circuit Theory, C.T. pp 248-259, 1959.

(156) Bogdanoff, J. L. and Kozin, F., "On the Moments of the Output of Linear Random Systems", J. Acoust. Soc. Amer. Vol. 34, p 1063, 1962.

(157) Bertram, J. E. and Sarachik, P. E., "Stability of Circuits with Randomly Time - Varying Parameters", Proc. Int. Symp. on Circuit and Inf. Theory, Los Angeles, Calif., I.R.E. Trans., C.T. 6, pp 260-270, 1959.

(158)  Kats, I. Ia. and Krasovskii, N. N., "On the Stability of Systems with Random Parameters", _PMM_, Vol. 24 , No. 5, pp 809-823, 1960.

(159)  Krasovskii, N. N., "On Meon - Square Optimum Stabilization at Damped Random Parameters", _PMM_ Vol. 25, pp 806-817, 1961.

(160)  Krasovskii, N. N. and Lidskii, E. A.., "Analytic Design of Controllers in Systems with Random Attributes.  Part I," _Avto. i Tele._, Vol. 22, No. 9, 1961.

(161)  Krasovskii, N. N. and Lidskii, E. A.. "Analytic Design of Controllers in Systems with Random Attributes, Part II." _Avto. i Tele._, Vol. 22, No. 10, pp 1273-1278, 1961.

(162)  Novoselov, V. S., "Investigation of the Stochastic Stability by the Example of Automatic Course - Line Control of the Aircraft", _Avto. i Tele._, Vol. 21, No. 6, 1960.

(163)  Lidskii, E. A., "On the Stabilization of Stochastic Systems", _PMM_, Vol. 25, No. 5, pp 824-835, 1961.

(164)  Khas'minskii, R. Z., "Ergodic Properties of Recurrent Diffusion Processes and Stabilization of the Solution to the Cauchy Problem for Parabolic Equations" _Theory of Prob. and It's Appl._, Vol. 5, No. 2, pp 179-196, 1960.

(165)  Khas' minskii, R. Z., "On the Stability of the Trajectory of Markov Processes", _PMM_, Vol. 26, No. 6, pp 1025-1032, 1962.

(166)  Kozin, F., "On Almost Sure Stability of Linear Systems with Random Coefficients", _Jour. of Math. and Physics_, Vol. 42, pp 59-67, 1963.

(167)  Caughey, T. K., "Comments on 'On the Stability of Random Systems'," _J. of the Acoust. Soc._, Vol. 32, p 1356, 1960.

(168)  Caughey, T. K. and Dienes, J. K., "The Behavior of Linear Systems with Random Parametric Excitation", _J. of Math. and Phys._, Vol. 41, pp 300-310, 1962.

(169)  Caughey, T. K. and Gray, A. H., Jr., "On the Almost Sure Stability of Linear Dynamic Systems With Stochastic Coefficients", _ASME - Appl. Mech._, Paper No. 64 - APM - 46, 1964.

(170)  Bharucha, B. H. "On the Stability of Randomly Varying Systems" <u>Elec. Res. Lab. Rep. Ser. 60 Issue No. 301</u>, Univ. of Cal., Berkeley, Calif., 1961.

(171)  Chelpanov, I. B., "Vibrator of a Second Order System with a Randomly Varying Parameter," <u>PMM</u>, Vol. 26, pp 1145 - 1152, 1962.

(172)  Gray, A. H., Jr., <u>Stability and Related Problems in Randomly Excited Systems</u>, Cal. Tech., Ph. D. Thesis, 1964.

(173)  Kats, I. Ia., "On the Stability of Stochastic Systems in the Large", <u>PMM</u>, Vol. 28, No. 2, pp 366-372, 1964.

(174)  Bershad, N. J., <u>Stability of Linear Systems with Gaussian Random Parameters</u>, Ph. D. Thesis, Rensselaer Polytechnic Institute, Tray, N. Y., June, 1962.

(175)  Tuel, W. G., <u>Topics on the Stability of Random Linear Systems</u>, M. S. Thesis, Rensselaer Polytechnic Institute, Troy, N. Y., Jan. 1964.

(176)  Bershad, N. J. and Derusso, P. M., "On the Moment Stability of Gaussian Random Linear Systems", <u>IEEE Trans. Auto. Control</u>, pp 301, 303, July, 1964.

(177)  Bershad, N. J., "On the Stability of Randomly Time - Varying Linear Systems", <u>IEEE Trans. Auto. Control</u>, p 319, July 1964.

(178)  Vrkoc, I., "On Some Stability Problems", <u>Diff. Eq. and Their Appl.</u>, Proc. of Prague Conf., Sept. 1962, Publ. by Academic Press, pp 217-228, 1963.

(179)  Kushner, H. J., <u>On the Theory of Stochastic</u> Stability, TR 65-1, Center for Dynamical Systems, Brown University, 1965.

(180)  Wonham, W. W., <u>On Weak Stochastic Stability of Systems Perturbed by Noise</u>, TR 65-4, Center for Dynamical Systems, Brown University, 1965.

(181)  Bucy, R. S., "Stability and Positive Supermartingales", <u>J. of Diff. Eq.</u>, Apr. 1965.

(182)  Zubov, V. I., <u>Methods of A. M. Lyapunov and Their Applications</u>,(Eng. Trans.), AEC - tr - 4439, 1961, (Russian Publ. Date - 1957).

(183)  Hahn, W., _Theory and Application of Liapunov's Direct Method_, (Eng. Trans.),

Prentice - Hall, Englewood Cliffs, N. J., 1963.

(184)  Movchan, A. A., "The Direct Method of Liapunov in Stability Problems of Elastic

Systems," _PMM_ Vol. 23, No. 3, pp 483-493, 1959.

(185)  Movchan, A. A., "On the Stability of Motion of Solid Bodies.  Lagrange's

Theorem and It's Inversion," _Inzh. sb. Izd-vo Akad. Nauk SSSR_, Vol. 29, 1960.

(186)  Movchan, A. A., "Stability of Processes with Respect to Two Metrics," _PMM_,

Vol. 24, No. 6, pp 988-1001, 1960.

(187)  Kostandian, B. A., "On the Stability of Solutions of the Nonlinear Equation

of Heat Conductor," _PMM_, Vol. 24, No. 6, pp 1112-1114, 1960.

(188)  Bellman, R., "On the Existence and Boundaries of Solutions of Nonlinear Partial

Differential Equations of Parabolic Type," _Trans. Amer. Math. Soc._ 64, 1948.

(189)  Mlak, W., "Remarks on the Stability Problem for Parabolic Equations" _Ann. polon._

_math._, Vol. 3, No. 3, 1957.

(190)  Rakhmatullina, L. F., "On the Problem of Stability of the Solution of the

Nonlinear Heat-Conductor Equation", _PMM_, Vol. 25, No. 3, pp 591-592, 1961.

(191)  Rakhmatullina, L. F., "On the Application of the Conditions Under Which

Chaplygin's Problem Possesses a Solution to Problems of Boundedness and Stability

of the Solutions of Differential Equations," _Izv. vuzov. Matematika_, No. 2, Vol. 9,

1959.

(192)  Slobodkin, A. M., "On the Stability of the Equilibrium of Conservative Systems

with an Infinite Number of Degrees of Freedom," _PMM_, Vol. 26, No. 2, pp 356-358, 1962.

(193)  Lakshmikantham, V., "Differential Systems and Extension of Lyapunov's Method,"

_Mich. Math. Jour._ Vol. 9, pp 311-320, 1962.

(194)  Lakshmikantham, V., "Differential Equations in Banach Spaces and Extension of

Lyapunov's Method," _Proc. Cambridge Phil. Soc._, Vol. 59, pp 373-381, 1963.

(195)   Lakshmikantham, V., "Functional Differential Systems and Extension of Lyapunov's Method," <u>J. Math. Anal. Appl.</u>,Vol. 9, 1964.

(196)   Lakshmikantham, V., "Parabolic Differential Equations and Lyapunov Like Functions,' <u>J. Math. Anal. Appl.</u>, Vol. 9, pp 234-251, 1964.

(197)   Krasovskii, N. N., <u>Stability of Motion</u>, (Eng. Trans.) Stanford Univ. Press, 1963.

(198)   Bellman, R. and Cooke, K. L., <u>Differential - Difference Equations</u>, Academic Press, New York, pp 334-392, 1963.

(199)   Bellman, R. and Danskin, J. M., Jr. "The Stability Theory of Differential-Differenc Equations", Rep. by the Rand Corp., 1953.

(200)   Razumikhin, B. S., "On the Stability of Systems with Lag", <u>PMM</u>, Vol. 20, No. 4, pp 500-512, 1956.

(201)   Razumikhin, B. S., "Stability in First Approximation of Systems with Lag," <u>PMM</u>, Vol. 22, No. 2, pp 155-166, 1958.

(202)   Glukhov, V. K., "Automatic Regulation of the Salt Content in Boiler Water", <u>Teploenergetika</u>, No. 7, 1954.

(203)   Shimanov, S. N., "On the Instability of the Motion of Systems with Retardation", <u>PMM</u>, Vol. 24, No. 1, pp 55-63, 1960.

(204)   Shimanov, S. N., "On the Stability in the Critical Case of a Zero Root for Systems with Time Lag", <u>PMM</u>, Vol. 24, No. 3, pp 447-457, 1960.

(205)   Popov, V. M. and Halanay, A.", On the Stability of Nonlinear Automatic Control Systems With Lagging Arguments", <u>Avtomatika i Telemekhanika</u>, Vol. 23, No. 7, pp 849-851, 1962.

(206)   Razumikhin, B. S., "Stability of Nonlinear Automatic Control Systems with Lag", <u>Inzhenemyisb</u> Vol. 29, 1960.

(207)   Hale, J., "Asymptotic Behavior of the Solutions of Differential-Difference Equations", <u>RIAS Rep. TR 61-10</u>, 1961.

(208) Driver, R. D., "Existence and Stability of Solutions of a Delay - Differential System", <u>Archiv. for Rat. Mech. and Anal.</u>, Vol. 10, pp 401-426, 1962.

(209) Wang, P.K.C. and Bandy, M.L.,"Stability of Distributed - parameter Processes with Time - Delays", <u>J. of Elec. and Control</u>, London, Eng., pp 343-362, 1963.

(210) Wang, P.K.C., "Asymptotic Stability of a Time - Delayed Diffusion System", <u>Trans. ASME, J. of Appl. Mech.</u>, pp 500-504, Dec., 1963.

(211) Hale, J., "Periodic and Almost Periodic Solutions of Functional-Differential Equations", <u>RIAS Rep. TR 63-4</u> 1963.

(212) Hale, J., "Linear Functional - Differential Equations with Constant Coefficients", <u>RIAS Rep. TR. 63-6</u>, 1963.

(213) Hale, J. "A Stability Theorem for Functional Differential Equations", <u>Proc. of the Nat. Acad. of Sc.</u>, Vol. 50, No. 5, pp 942-946, 1963.

(214) LaSalle, J. P., "The Extent of Asymptotic Stability", <u>Proc. of the Nat. Acad. of Sc.</u>, Vol. 46, pp 363-365, 1960.

(215) Hale, J. K., "Functional-Differential Equations with Parameters", <u>Contr. to Diff. Eq.</u> Vol. 1, No. 4, pp 401-410, 1963.

(216) Hale, J., "Asymptotic Behavior of the Solutions of Differential-Difference Equations", <u>Proc. Symp. on Nonlinear Oscillations</u>, IUTAM, Kiev, USSR, 1961.

(217) Hale, J., "A Class of Functional-Differential Equations," <u>Contr. of Diff. Eq.</u>, Vol. 1, No. 4, pp 411-423, 1963.

(218) Seifert, G., "Stability Conditions for Separation and Almost Periodcity of Solutions of Differential Equations", <u>Contr. to Diff. Eq.</u>, Vol. 1, No. 4, pp 483-487, 1963.

(219) Seifert, G., "Uniform Stability of Almost-Periodic Solutions of Almost-Periodic Systems of Differential Equations", <u>Contr. to Diff. Eq.</u>, Vol. II, pp 269-276, 1963.

(220) Stokes, A., "The Applications of a Fixed Point Theorem to a Variety of Non-Linear Stability Problems", <u>Contr. to the Theory of Nonlinear Oscillations</u>, Vol. V, Princeton

University Press, pp 173-184, 1960.

(221)   Jones, G. S., "Asymptotic Fixed Point Theorems and Periodic Systems of Functional - Differential Equations", <u>Contr. to Diff. Eq.</u>, Vol. II, pp 385-405, 1963.

(222)   Taam, C. T., "Existence and Stability of Almost Periodic Solutions of Differential Equations in Banach Spaces," <u>Math. Res. Center Tech. Summary Rep #455</u>, University of Wisconsin, 1964.

(223)   Jones, G. S., "Periodic Motions in Banach Space and Applications to Functional-Differential Equations," <u>Contr. to Diff. Eq.</u> Vol. III, No. 1, pp 75-106, 1964.

(224)   Stokes, A., "On the Stability of a Limit Cycle of An Autonomous Functional-Differential Equations", <u>Contr. to Diff. Eq.</u>, Vol. III, No. 2, pp 121-139, 1964.

(225)   Hale, J. and Perello, C., "The Neighborhood of a Singular Point of Functional Differential Equations", <u>Contr. to Diff. Eq.</u>, Vol. III, No. 3, 1964.

(226)   Hale, J., "Sufficient Conditions for Stability and Instability of Autonomous Functional - Differential Equations," <u>Res. Rep. written under the support of NASA, U. S. Army, U. S. Air Force</u>, 1965.

(227)   Nemyckii, V. V., <u>Topological Problems of the Theory of Dynamical Systems</u>, Amer. Math. Soc. Trans. No. 103, 1949.

(228)   LaSalle, J. P., "A Study of Synchnonous Asymptotic Stability", <u>Annals of Math.</u>, Vol. 65, No. 3, 1957.

(229)   Lefschetz, S., "Lyapunov and Stability in Dynamical Systems", <u>RIAS Rep. TR 58-2</u> 1958.

(230)   Peixoto, M. M., "Structural Stability on Two - Dimensional Manifolds", <u>Inst. de Mate Pura e Apl.</u> 1961.

(231)   Ura, T., "Stability of a Closed Invariant Set and its Nearby Lying Invariant Sets", <u>RIAS Rep. TR 62-14</u>, 1962.

(232)   Auslander, J. and Seibert, P., "Prolongation and Stability in Dynamic Systems", <u>RIAS Rep. TR63-18</u>, 1963.

(233) Hale, J. and Onuchic, N., "On the Asymptotic Behavior of Solutions of a Class of Differential Equations", <u>Contr. to Diff. Eq.</u>, Vol. II, pp 61-75, 1963.

(234) Auslander, J., "Generalized Recurrence in Dynamical Systems", <u>Contr. to Diff. Eq.</u> Vol. III, No. 1, pp 65-74, 1964.

(235) Bhatia, N. P., "Stability and Liapunov Functions in Dynamic Systems", <u>Contr. to Diff. Eq.</u>, Vol. III, No. 2, pp 175-188, 1964.

(236) Ura, T., "On the Flow Outside a Closed Invariant Set; Stability, Relative Stability and Saddle Sets", <u>Contr. to Diff. Eq.</u> Vol. III, No. 3, pp 249-294, 1964.

(237) Pugh, C. C., "The Closing Lemnia and Structural Stability", <u>Bull. Amer. Math. Soc.</u> Vol. 70, No. 4, pp 584-587, 1964.

(238) Jenkins, T. R., Johnson, W. E. and Petty, C. M., "Contributions to the Qualitative Theory of Differential Equations, (Topological Dynamics), Vol. I.", <u>Lockheed Indep. Res. Program Rep.</u> TR-6-74-64-44, 1964.

(239) Jenkins, T. R., Johnson, W. E. and Petty, C. M., "Contributions to the Quatitative Theory of Differential Equations, (Topo. Dyn.), Vol. I.", <u>Lockheed, Missiles and Space Company</u> Rep., 1964.